CONIC SECTIONS,

WITH

SELECT EXERCISES

IN VARIOUS BRANCHES OF

MATHEMATICS AND PHILOSOPHY.

FOR THE USE OF THE ROYAL MILITARY
ACADEMY AT WOOLWICH.

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TO HIS GRACE

CHARLES,

DUKE OF RICHMOND, LENNOX, AND AUBIGNY,
&c. &c. &c.

MASTER GENERAL OF THE ORDNANCE.

MY LORD,

SHOULD this ſmall performance continue to prove uſeful to the Inſtitution for which it was compoſed, or become acceptable to the public; its exiſtence, as well as its publication, muſt be aſcribed to YOUR GRACE.

But, my LORD, the particular injunctions to avoid panegyric, under which permiſſion for this dedication was granted, preclude me from explaining how the advanced ſtate of learning in the Royal Military Academy rendered

dered ſuch a work neceſſary: ſince this would be to enumerate the means by which, under Your Auſpices, this inſtitution has attained a degree of perfection, which perhaps few public ones have ever equalled, none certainly have exceeded.

To YOUR GRACE therefore, with all Reſpect, this work is moſt humbly dedicated by,

MY LORD,

YOUR GRACE'S

moſt obedient,

moſt devoted, and

moſt humble ſervant,

CHARLES HUTTON.

Royal Mil. Acad.
Aug. 24, 1787.

PREFACE.

THE want of a proper ſet of exerciſes, applied to the different branches of mathematical knowledge, which are deemed requiſite to the military profeſſion, induced me to draw up the following ſheets. Their utility to the ſtudents in the Royal Military Academy, having been fully eſtabliſhed, recommended them to the conſideration of the Maſter General; and His Grace has been pleaſed to order them to be printed.

The miſcellaneous form of this ſmall work, ariſes from its conſiſting chiefly of practical queſtions in moſt of the ſciences now taught in the Academy. Although Invention was not my immediate object, yet throughout the whole there will be found many things that are new, in *point of matter, but more ſo in the manner of treating the ſubjects.*

In the Conic Sections there are ſeveral new and important properties diſperſed through the books; and it is preſumed that this branch is treated in a way better adapted to its intended uſe, than heretofore. The propoſitions, although demonſtrated in a manner ſtrictly geometrical, have this peculiarity, that only the firſt property of each ſection is demonſtrated from the cone itſelf, and all the ſubſequent ones derived from the firſt, or from each other, in an eaſy and natural way, without introducing any arbitrary organical deſcription of curves in plano. The arrangement of the ſteps in ſeparate lines is alſo found to be an advantage, as it renders the demonſtrations more eaſy to be comprehended, by preſenting the whole to the eye in one connected view. And another very conſiderable improvement ariſes from the application of a new and general property concerning the interſections of a right line, with any of the curves, in two points: by means of which the general properties of the oblique ordinates, to any diameter, are eaſily deduced, without the forced and perplexed conſideration of the areas of certain ſpaces.

The ſeveral propoſitions and properties, in the three curves or ſections, are alſo arranged in ſuch order, and enunciated in ſuch manner, as to ſhew *which properties are common to the different* ſections; and in particular it will be found, that moſt of the definitions and ſcholia are common to all the three curves; and that all the propoſitions and demonſtrations of the ellipſe, are literally the ſame with thoſe of the hyperbola: a circumſtance which muſt render both the learning and the remembering of the properties much eaſier than heretofore.

The collection of practical queſtions, which follow the Conic Sections, are moſtly given without ſolutions, their anſwers only being ſet down, as probationary exerciſes to certain rules and branches of ſcience contained in moſt books relating to theſe ſubjects. But the laſt collection, concerning forces, and the accompanying circumſtances of time, ſpace, and the velocity generated, have ſolutions annexed to them; as they require a knowledge of ſome other principles beſides thoſe that are uſually found in the common books of ſcience. Many of the problems in this part may, indeed, be met with elſewhere; but it is preſumed, that the ſolutions will be found in general, either new, or attended with conſiderable improvement.

I have taken the liberty alſo to enrich this part with ſome new and uſeful problems relating to the times of filling and emptying the ditches of fortifications, or other receptacles, with water, entering and evacuating them under certain circumſtances. Theſe curious problems, His Grace the Maſter General of the Ordnance was pleaſed to propoſe at a late examination of the Gentlemen Cadets; and the ſolutions at large, of ſuch important propoſitions, are here publiſhed, as far as I know, for the firſt time.

To theſe ſucceeds the common theory of the motion of bodies in reſiſting mediums, but delivered in a manner, I truſt, better adapted to practical purpoſes than in any former publication.

The volume then concludes with a compendium of ſome experiments, lately made to aſcertain the actual reſiſtance of the air to given ſurfaces, moving through it with given velocities, and different degrees of inclination: experiments, which it is to be wiſhed may be farther proſecuted, as it is by ſuch means only that the true theory of military projectiles, as well as other branches of natural philoſophy, can be improved to any degree of practical utility.

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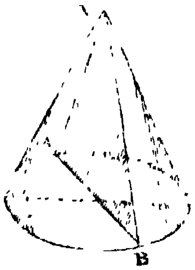
CONIC SECTIONS.

DEFINITIONS.

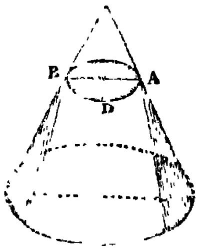
1. CONIC ſections are the figures made by the mutual interſection of a cone and a plane.

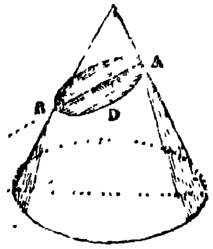
2. According to the different poſitions of the cutting plane, there ariſe five different figures or ſections, namely, a triangle, a circle, an ellipſe, a parabola, and an hyperbola: the three laſt of which only are peculiarly called conic ſections.

3. If the cutting plane paſs through the vertex of the cone, and any part of the baſe, the ſection will evidently be a triangle; as VAB.

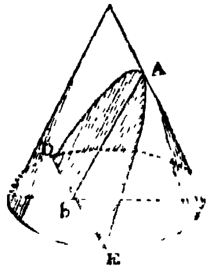


4. If the plane cut the cone parallel to the baſe, or make no angle with it, the ſection will be a circle; as ABD.

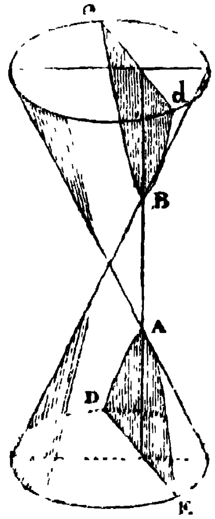




5. The ſection DAB is an ellipſe, when the cone is cut obliquely through both ſides, or when the plane is inclined to the baſe in a leſs angle than the ſide of the cone is.



6. The ſection is a parabola, when the cone is cut by a plane parallel to the baſe, or when the cutting plane and the ſide of the cone make equal angles with the baſe.



7. The ſection is an hyperbola, when the cutting plane makes a greater angle with the baſe than the ſide of the cone makes.

8. And if all the ſides of the cone be continued through the vertex, forming an oppoſite equal cone, and the plane be alſo continued to cut the oppoſite cone, this latter ſection will be the oppoſite hyperbola to the former ; as dBe.

9. The vertices of any ſection, are the points where the cutting plane meets the oppoſite ſides of the cone, or the ſides of the vertical triangular ſection ; as A and B.

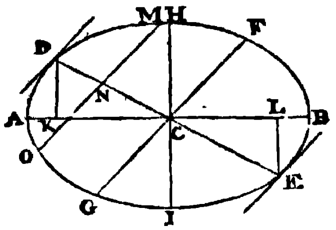
Hence the ellipſe and the oppoſite hyperbolas, have each two vertices ; but the parabola only one ; unleſs we conſider the other as at an infinite diſtance.

10. The Axis, or Tranſverſe Diameter, of a conic ſection, is the line or diſtance AB between the vertices.

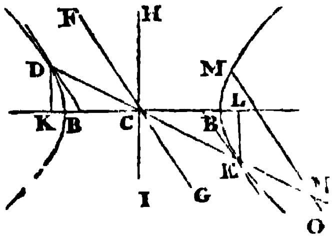
Hence the axis of a parabola is infinite in length, Ab being only a part of it.

11. The

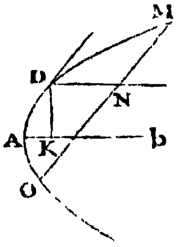
Ellipſe.



Oppos. Hyperb.



Parabola.



11. The Center C is the middle of the axis.

Hence the center of a parabola is infinitely diſtant from the vertex. And of an ellipſe the axis and center lie within the curve; but of an hyperbola without.

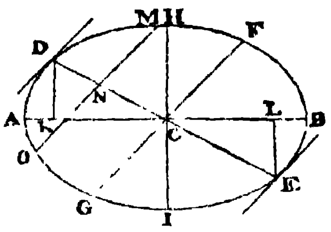
12. A Diameter is any right line, as AB or DE, drawn through the center, and terminated on each ſide by the curve; and the extremities of the diameter, or its interſections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence alſo every diameter of the ellipſe and hyperbola have two vertices; but of the parabola only one; unleſs we conſider the other as at an infinite diſtance.

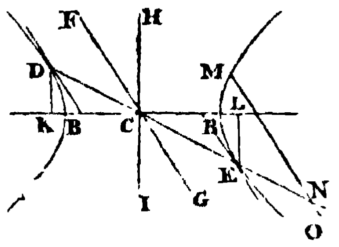
13. The Conjugate to any diameter, is the line drawn through the center, and parallel to the tangent of the curve at the vertex of the diameter. So FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

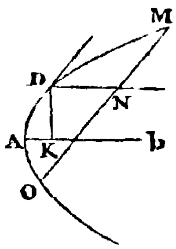
Ellipſe.



Oppos. Hyperb.



Parabola.



14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL are ordinates to the axis AB; and MN, NO ordinates to the diameter DE.

Hence the ordinates to the axis are perpendicular to it.

15. An Abſciſs is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

Hence, in the ellipſe and hyperbola, every ordinate has two abſciſſes; but in the parabola, only one; the other vertex of the diameter being infinitely diſtant.

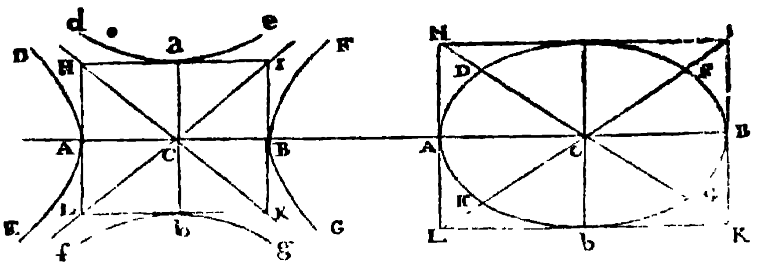
16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the ſemi-parameter.

Hence, the ellipſe and hyperbola have each two foci; but the parabola only one.

18. If DAE, FBG be two oppoſite hyperbolas having AB for their firſt or tranſverſe axis, and ab for their ſecond or

con-



conjugate axis. And if dae, fbg be two other oppoſite hyperbolas having the ſame axes, but in the contrary order, namely, ab their firſt axis, and AB their ſecond; then theſe two latter curves dae, fbg, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of oppoſite curves mutually conjugate to the other.

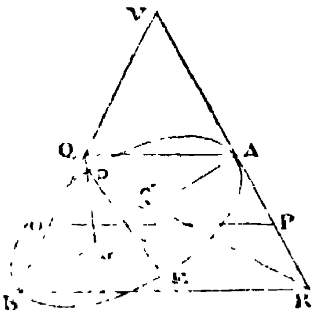
19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inſcribed rectangle HIKL; the diagonals HCK, ICL of this rectangle, are called the aſymptotes of the curves.

SCHOLIUM.

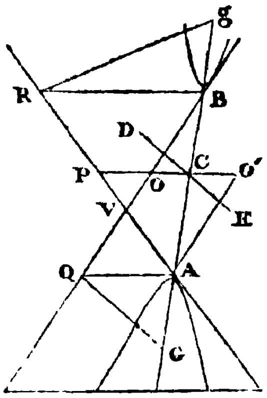
The rectangle inſcribed between the four conjugate hyperbolas, is ſimilar to a rectangle circumſcribed about an ellipſe by drawing tangents, in like manner, to the four extremities of the two axes; and the aſymptotes or diagonals in the hyperbola, are analogous to thoſe in the ellipſe, cutting this curve in ſimilar points, and making the pair of equal conjugate diameters. Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipſe turned inſide out, cut open at the extremities D, E, F, G, of the ſaid equal conjugate diameters, and thoſe four points drawn out to an infinite diſtance, the curvature being turned the contrary way, but the axes, and the rectangle paſſing through their extremities, continuing fixed.

COROLLARY I.

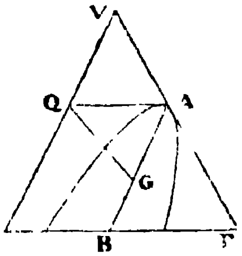
Ellipſe.



Hyperbola.



Parabola.



In the ellipſe, the ſemi-conjugate axis, CD or CE, is a mean proportional between CO and CP, the parts of the diameter OP of a circle drawn through the center C of the ellipſe, and parallel to the baſe of the cone. For DE is a double ordinate in this circle, being perpendicular to OP as well as to AB.

In like manner, in the hyperbola, the length of the ſemi-conjugate axis, CD or CE, is a mean proportional between CO and CP, drawn parallel to the baſe, and meeting the ſides of the cone in O and P. Or, if AO' be drawn parallel to the ſide VB, and meet PC produced in O', making $CO' = CO$; and on this diameter $O'P$ a circle be drawn parallel to the baſe: then the ſemi-conjugate CD or CE will be an ordinate of this circle, being perpendicular to $O'P$ as well as to AB.

Or, in both figures, the whole conjugate axis DE is a mean proportional between QA and BR, parallel to the baſe of the cone. For, becauſe AB is double of AC or CB, therefore, by ſimilar triangles, QA is double of OC, and BR double of CP; conſequently

DE^2 or $2C \cdot 2E$, or $2CO \cdot 2CP$ is $= QA \cdot BR$, or
$QA : DE :: DE : BR$.

In the parabola both the tranſverſe and conjugate are infinite; for AB and BR are both infinite.

CoROL. 2. In all the ſections AG will be equal to the parameter of the axis, if QG be drawn making the angle AQG equal to the angle BAR.

For, by the definition, AB : DE :: DE : p the param.
But by corol. 1, BR : DE :: DE : AQ;
Therefore AB : BR :: AQ : p.
But, by ſimilar triangles, AB : BR :: AQ : AG;
And therefore AG $=$ p the parameter.

In like manner Bg will be equal to the parameter p, if Rg be drawn to make the angle BRg $=$ the angle ABQ; ſince here alſo AB : AQ :: BR : BG $=$ p.

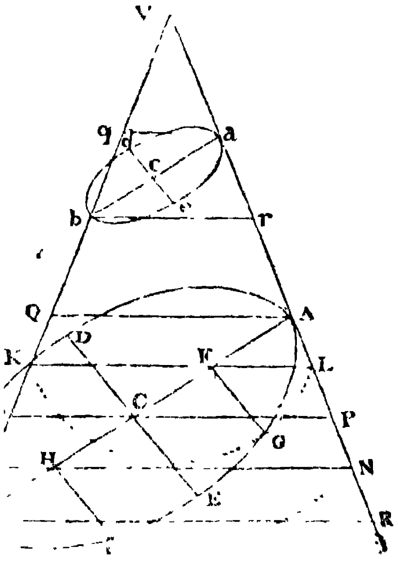
CoROL. 3. Hence the upper hyperbolic ſection, or ſection of the oppoſite cone, is equal and ſimilar to the lower ſection. For the two ſections have the ſame tranſverſe or firſt axis AB, and the ſame conjugate or ſecond axis DE, which is the mean proportional between AQ and RB; they have alſo equal parameters AG, Bg. So that the two oppoſite ſections make, as it were, but the two oppoſite ends of one entire ſection or hyperbola, the two being every where mutually equal and ſimilar. Like the two halves of an ellipſe, with their ends turned the contrary way.

CoROL. 4. And hence, although both the tranſverſe and conjugate axis in the parabola be infinite, yet the former is infinitely greater than the latter, or has an infinite ratio to it. For the tranſverſe has the ſame ratio to the conjugate, as the conjugate has to the parameter, that is, as an infinite to a finite quantity, which is an infinite ratio.

OF THE ELLIPSE.

PROPOSITION I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abſciſſes.



Let AVB be a plane paſſing through the axis of the cone; AGIH another ſection of the cone perpendicular to the plane of the former; AB the axis of this elliptic ſection; and FG, HI ordinates perpendicular to it. Then I ſay that

$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB.$

For, through the ordinates FG, HI draw the circular ſections KGL, MIN parallel to the baſe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the ellipſe.

Now, by the ſimilar triangles AFL, AHN, and BFL, BHM, we have $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

hence, taking the rectangles of the correſponding terms, we have the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the nature of the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;

Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q.E.D.

Corol. 1. All the parallel ſections are ſimilar figures, or have their two axes in the ſame proportion; that is, $AB : ab :: DE : de$.

For, by ſim. triang. $AB : ab :: AQ : aq$,
and $AB : ab :: RB : rb$;
Theref. by comp. $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.
But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;
Therefore $AB^2 : ab^2 :: DE^2 : de^2$,
or $AB : ab :: DE : de$.

Corol. 2. Hence alſo, as the property is the ſame for the ordinates on both ſides of the diameter, it follows, that

1ſt. At equal diſtances from the center, or from the vertices, the ordinates on both ſides are equal, or that the double ordinates are biſected by the axis; and that the whole figure, made up of all the double ordinates, is alſo biſected by the axis.

2d. The two foci are equally diſtant from the center, or from either vertex.

Corol. 3. When the angle, which the plane of the ſection makes with the baſe of the cone, increaſes till it become equal to the angle made by the ſide of the cone and the baſe, or till the ſection be parallel to the oppoſite ſide of the baſe; then the axis becomes infinitely long, and the ellipſe degenerates into a parabola; and becauſe then the infinites FB and HB are in a ratio of equality, the general property,

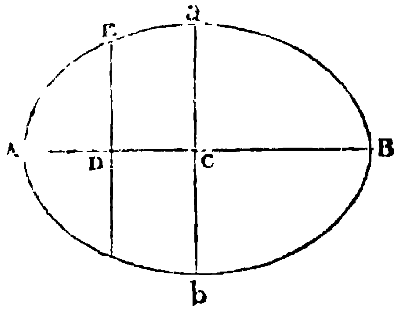
namely $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,
becomes $AF : AH :: FG^2 : HI^2$,

or, in the parabola, the abſciſſes are to each other, [illegible] the ſquares of their ordinates.

PROPOSITION II.

As the Square of the Tranſverſe Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abſciſſes :
To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$.



For, by prop. 1. $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
But, if C be the center, then $AC \cdot CB = AC^2$, and ca is the ſemi-conj.
Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q.E.D.

Corol. 1. Or, becauſe the rectangle $AD \cdot DB = CA^2 - CD^2$, the ſame property is $CA^2 : ca^2 :: CA^2 - CD^2 : DE^2$, or $AB^2 : ab^2 :: CA^2 - CD^2 : DE^2$.

Corol. 2. Or, by div. $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$, that is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$; where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the tranſverſe,
Is to its parameter,
So is the rectangle of the abſciſſes,
To the ſquare of their ordinate.

Corol. 3. When the axis AB is infinitely long, the curve becomes a parabola, and the infinites AB, DB are then in a ratio of equality; and then the laſt property,

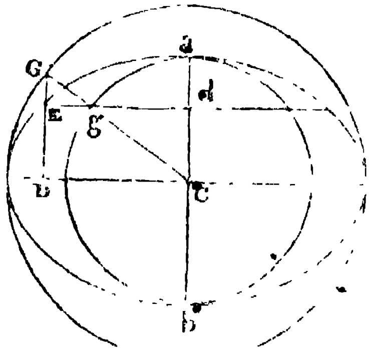
namely $AB : p :: AD \cdot DB : DE^2$;
or $AB \cdot DE : AD \cdot DB :: p : DE$,
becomes $DE : AD :: p : DE$,
or $AD : DE :: DE : p$.

That is, in the parabola, the parameter is a third proportional to any abſciſs and its ordinate.

PROPOSITION III.

As the Square of the Conjugate Axis:
Is to the Square of the Tranſverſe Axis ::
So is the Rectangle of the Abſciſſes of the Conjugate, or the Difference of the Squares of the Semi-conjugate and Diſtance of the Center from any Ordinate of that Axis:
To the Square of that Ordinate.

That is, $ca^2 : CB^2 :: ad \cdot db$ or $ca^2 - cd^2 : dE^2$.



For draw the ordinate ED to the tranſverſe AB.

Then, by cor. 1. prop. 2. $CA^2 : ca^2 :: CA^2 - CD^2 : DE^2$.
But $CD^2 = dE^2$, and $DE^2 = cd^2$,
therefore $CA^2 : ca^2 :: CA^2 - dE^2 : cd^2$,
or by alternation, $CA^2 : CA^2 - dE^2 :: ca^2 : cd^2$,
and by diviſion, $CA^2 : dE^2 :: ca^2 : ca^2 - cd^2$,
and by alter. & inverſ. $ca^2 : CA^2 :: ca^2 - cd^2 : dE^2$. Q.E.D.

Corol. 1. *If two circles be deſcribed on the two axes* as diameters, the one inſcribed within the ellipſe, and the other circumſcribed about it; then an ordinate in the circle will be to the correſponding ordinate in the ellipſe, as the axis of this ordinate, is to the other axis.

That is, $CA : Ca :: DG : DE$,
and $ca : CA :: dg : dE$.

For, by the nature of the circle, $AD \cdot DB = DG^2$; theref. by the nature of the ellipſe, $CA^2 : Ca^2 :: AD \cdot DB$ or $DG^2 : DE^2$,
or $CA : Ca :: DG : DE$.
In like manner $ca : cA :: dg : dE$.
Moreover, by equality, $DG : DE$ or $cd :: dE$ or $DC : dg$.

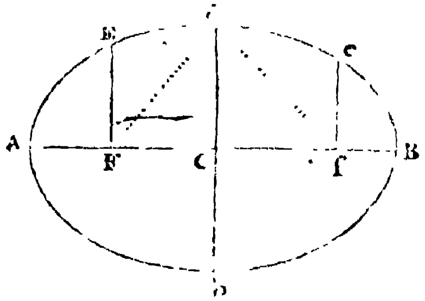
Therefore cgG is a continued ſtrait line.

Corol. 2. Hence alſo, as the ellipſe and circle are made up of the ſame number of correſponding ordinates, which are all in the ſame proportion of the two axes, it follows that the areas of the whole circle and ellipſe, as alſo of any like parts of them, are in the ſame proportion of the two axes, or as the ſquare of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipſe, are as the ſquare of each axis and the rectangle of the two, and therefore the ellipſe is a mean proportional between the two circles.

PROPOSITION IV.

The Square of the Diſtance of the Focus from the Center, is equal to the Difference of the Squares of the Semi-axis; Or, the Square of the Diſtance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, $CF^2 = CA^2 - Ca^2$,
or $Ff^2 = AB^2 - ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the ſemi-parameter. Then by the nature of the curve $CA^2 : Ca^2 :: CA^2 - CF^2 : FE^2$;
and by the def. of the para. $CA^2 : Ca^2 :: Ca^2 : FE^2$;
therefore $Ca^2 = CA^2 - CF^2$;
and by addit. and ſubtr. $CF^2 = CA^2 - Ca^2$;
or, by doubling, $Ff^2 = AB^2 - ab^2$. Q.E.D.

Corol. 1. The two ſemi-axes, and the focal diſtance from the center, are the ſides of a right angled triangle CFa; and the diſtance Fa from the focus to the extremity of the conjugate axis, is = AC the ſemi-tranſverſe.

For, as above, $CA^2 - ca^2 = CF^2$,
and by right angled triangles $Fa^2 - ca^2 = CF^2$,
therefore $CA = Fa$, and $AB = Fa + fa$.

Corol. 2. The conjugate ſemi-axis ca is a mean proportional between AF, FB, or between Af, fB, the diſtances of either focus from the two vertices.

For $ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB$.

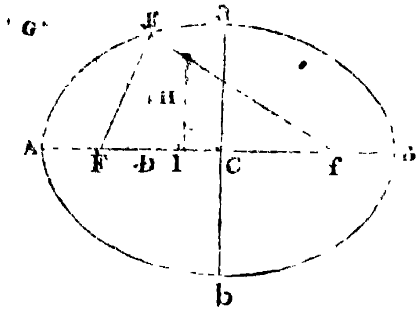
Corol. 3. The ſame rectangle $AF \cdot FB$ of the focal diſtances from either vertex, is alſo equal to the rectangle $AC \cdot FE$ under the ſemi-tranſverſe and its ſemi-parameter; ſince this laſt is equal to the ſquare of the ſemi-conjugate by the definition of the parameter.

Or $AF : FE :: AC : FB$.

PROPOSITION V.

The Difference between the Semi-tranſverſe and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-tranſverſe, the Diſtance from the Center to the Focus, and the Diſtance from the Center to the Ordinate belonging to that Point of the Curve.

That is, $AC - FE = CI$, or $FE = AI$;
and $fE - AC = CI$, or $fE = BI$.
Where $CA : CF :: CD : CI$ the 4th proportional to CA, CF, CD.



For, by right angled triangles, $FE^2 = FD^2 + DE^2$.

Now draw AG parallel and equal to Ca the ſemi-conjugate; and join CG meeting the ordinate DE in H.

Then, by prop. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by ſim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
conſequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.
Alſo $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But by prop. 4. $Ca^2 + CF^2 = CA^2$
and, by ſuppoſition, $2CF\cdot CD = 2CA\cdot CI$;
theref. $FE^2 = CA^2 - 2CA\cdot CI + CD^2 - DH^2$.

But, by ſuppoſition $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by ſim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore $CI^2 = CD^2 - DH^2$;
conſequently $FE^2 = CA^2 - 2CA\cdot CI + CI^2$.
And the root or ſide of this ſquare is $FE = CA - CI = AI$.

In the ſame manner is found $fE = CA + CI = BI$. Q.E.D.

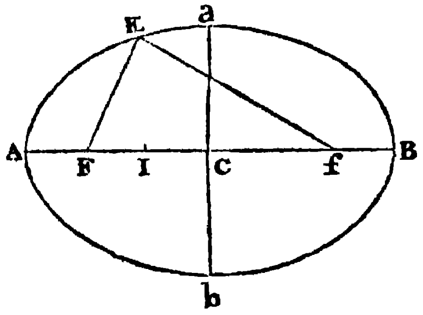
Corol. 1. Hence CI or $CA - FE$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE - FE = 2CI$; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

PROPOSITION VI.

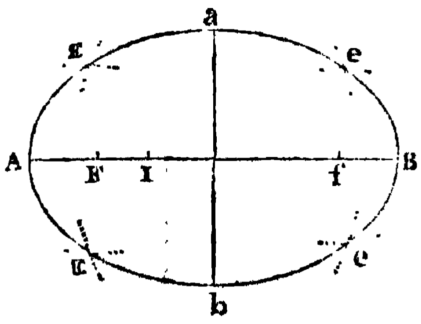
The Sum of two Lines drawn from the Foci, to meet in any Point of the Curve is equal to the Tranſverſe Axis.

That is, $FE + fE = AB$.



For, by the laſt prop. $FE = CA - CI = AI$,
and, by the ſame, $fE = CA + CI = BI$;
theref. by addition, $FE + fe = AB$.

Corol. Hence is derived the common method of deſcrib[illegible]the curve mechanically by points, or with a thread, thus.



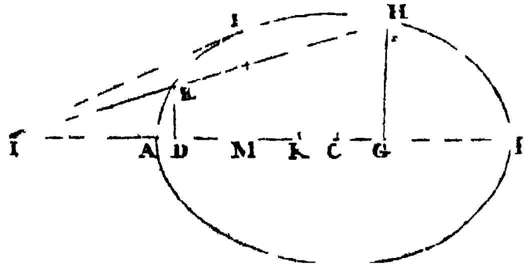
In the tranſverſe take the foci F, f, and any point I. Then with the radii AI, BI, and centers F, f, deſcribe arcs interſecting in E, which will be a point in the curve. In like manner, aſſuming other points I, as many other points will be found in the curve. Then with a ſteady hand, draw the curve line through all the points of interſection E.

Or, take a thread of the length of AB the tranſverſe axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always ſtretched, and its point will trace out the curve line.

PROPOSITION VII.

If from any Point I in the Axis produced, a Line IEH be drawn cutting the curve in Two Points; and from thoſe Two Points be drawn the Perpendicular Ordinates DE, GH; and if K be the Middle of DG, and C the Center or the Middle of AB: Then ſhall CK be to CI as the Rectangle of AD and AG to the Square of AI.

That is, $CK : CI :: AD \cdot AG : AI^2$.



For, by prop. I. $AD \cdot DB : AG \cdot GB :: DE^2 . GH^2$,

and by ſim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$;

theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = 2CK + AG$, and $GB = 2CK + AD$,

theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,

and, by div. $DG . 2CK \quad IG^2 - ID^2$ or $DG^2 . 2IK . AD . 2CK + AD . AG \quad ID^2$.

or $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$

or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;

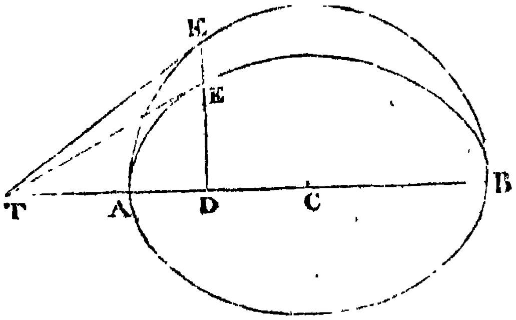
theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK$,
and, by comp. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or $CK : CI :: AD \cdot AG : AI^2$. Q.E.D.

Corol. When the line IH, by revolving about the point I, comes into the poſition of the tangent IL, and the ordinate LM being drawn, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the property in the propoſition becomes $CM : CI :: AM^2 : AI^2$.

PROPOSITION VIII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Center.

That is, CA is a mean proportion between CD and CT; or CD, CA, CT are continued proportionals.



For, by cor. prop. 7, $CD : CT :: AD^2 : AT^2$,

that is, $CD : CT :: \overline{CA - CD}^2 : \overline{CT - CA}^2$,

or $:: CD^2 + CA^2 : CA^2 + CT^2$,

and $:: CD^2 : CA^2$,

and $:: CA^2 : CT^2$;

therefore $CD : CA :: CA : CT$. Q.E.D.

Corol. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from E, of every ellipſe deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

COROL. 2. Hence a tangent is eaſily drawn to the curve, from any point, either in the curve or without it.

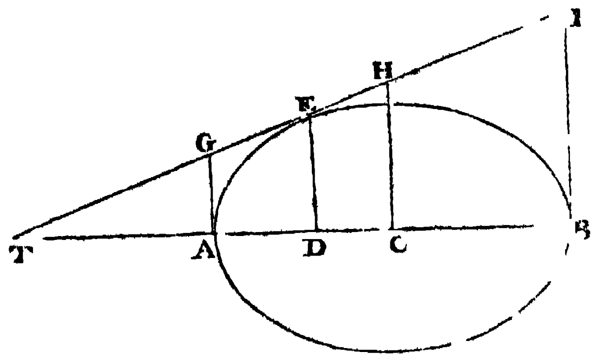
Firſt, if the given point E be in the curve. Draw the ordinate DE of the diameter AC; and in the diameter produced take CT a third proportional to CD, CA. Then join TE for the tangent required.

But if the point T be given any where without the curve. Join CT, in which take CD a third proportional to CT, CA; and draw the ordinate DE. Then join TE as before.

PROPOSITION IX.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four Points, namely the Center, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That is, AG : DE :: CH : BI.



For, by prop. 8, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by ſim. tri. AG . DE :: CH : BI. Q.E.D.

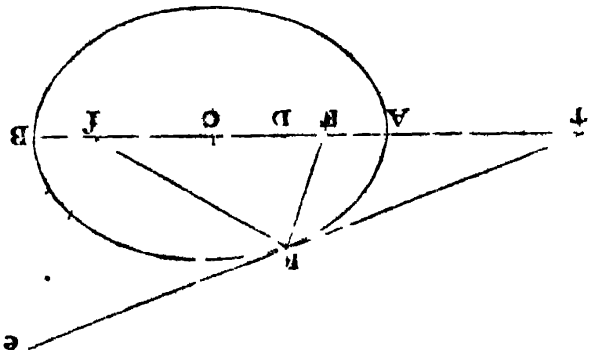
Corol. Hence TA, TD, TC, TB
and TG, TE, TH, TI } are alſo proportionals.

For theſe are as AG, DE, CH, BI, by ſimilar triangles.

PROPOSITION X.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; theſe two Lines will make equal Angles with the Tangent.

That is, the ∠ FET = ∠ fEe.



For draw the ordinate DE, and fe parallel to FE.

By cor. 1. prop. 5, CA : CD :: CF : CA — FE,
and by prop. 8, CA : CD :: CT : CA;
therefore CT : CF :: CA : CA — FE;
and by add. and ſub. TF : Tf :: FE . 2CA — FE or fE by prop. 6.
But by ſim. tri. TF : Tf :: FE : fe;
therefore fE = fe, and conſeq. ∠ e = ∠ fEe.
But, becauſe FE is parallel to fe, the ∠ e = ∠ FET;
therefore the ∠ FET = ∠ fEe. Q.E.D.

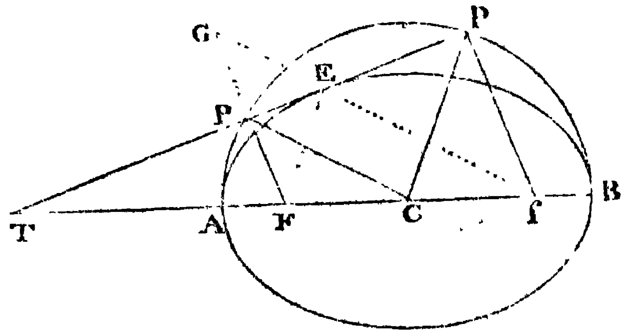
Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from our propoſition, that rays of light iſſuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reaſon why the points F, f are called *foci*, or burning points.

PRO-

PROPOSITION XI.

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve; the Diſtance of their Interſection from the Center will be equal to the Semi-tranſverſe Axis.

That is, if FP, fp be perpendicular to the Tangent TPp, then ſhall CP and cp be each equal to CA or CB.



For, through the point of contact E draw FE and fE meeting FP produced in G. Then, the ∠ GEP = ∠ FEP, being each equal to the ∠ fEp, and the angles at P being right, and the ſide PE being common, the two triangles GEP, FEP are equal in all reſpects, and ſo GE = FE, and GP = FP. Therefore, ſince FP = $\frac{1}{2}$ FG, and FC = $\frac{1}{2}$ Ff, and the angle at F common, the ſide CP will be or $\frac{1}{2}$AB, that is CP = CA or CB.

In the ſame manner cp = CA or CB. Q.E.D.

Corol. 1. A circle deſcribed on the tranſverſe axis, as a diameter, will paſs through the points P, p; becauſe all the lines CA, CP, Cp, CB, being equal, will be radii of the circle.

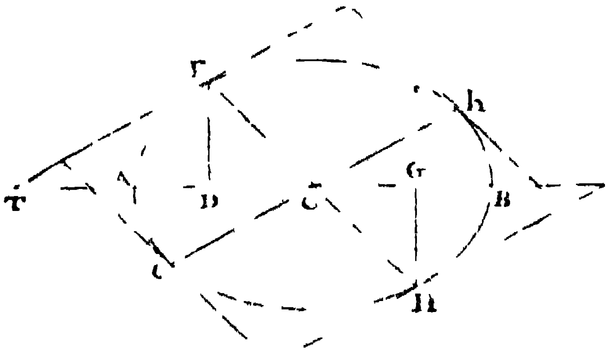
Corol. 2. CP is parallel to fE, and Cp parallel to FE.

Corol. 3. If at the interſections of any tangent, with the circumſcribed circle, perpendiculars to the tangent be drawn, they will meet the tranſverſe axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f.

PROPOSITION XII.

The equal Ordinates, or the Ordinates at equal Diſtances from the Center, on the oppoſite Sides and Ends of an Ellipſe, have their Extremities connected by one Right Line paſſing through the Center, and that Line is biſected by the Center.

That is, if CD = CG, or the ordinate DE = GH;
then ſhall CE = CH, and ECH will be a right line.



For, when CD = CG, then alſo is DE = GH by cor. 2. prop.
But the ∠D = ∠G, being both right angles;
therefore the third ſide CE = CH, and the ∠ DCE = ∠ GCH
and conſequently ECH is a right line.

COROL. 1. And, converſely, if ECH be a right line paſſing through the center; then ſhall it be biſected by the center, or have CE = CH; alſo DE will be = GH, and CD = CG.

Corol. 2. Hence alſo, if two tangents be drawn to the two ends E, H of any diameter EH they will be parallel to each other, and will cut the axis at equal angles, and at equal diſtances from the center. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal alſo; then the two ſides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other correſponding parts are equal: and ſo the ∠ T = ∠ S, and TE parallel to HS.

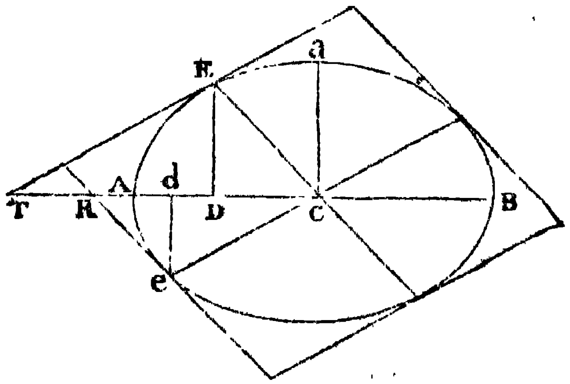
Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumſcribing the ellipſe, and the pairs of oppoſite ſides are each equal to the correſponding parallel conjugate diameters.

For, if the diameter eh be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to eh will be parallel to each other, and to the diameter EH for the ſame reaſon.

PROPOSITION XIII.

If two Ordinates ED, ed be drawn from the Extremities E, e, of two Conjugate Diameters, and Tangents be drawn to the ſame Extremities, and meeting the Axis produced in T and R;

Then ſhall CD be a mean Proportional between cd, dR, and cd a mean Proportional between CD, DT.



For, by prop. 8, CD : CA : CA : CT,
and by the ſame cd : CA : CA : CR;
theref. by equality CD : cd : CR : CT,
But by ſim. tri. DT : cd : CT : CR;
theref. by equality CD : cd : cd : DT.
In like manner cd : CD : CD : dR. Q.E.D.

Corol. 1. Hence CD : cd :: CR : CT.

Corol. 2. Hence alſo CD : cd :: de : DE.
And the rect. $CD \cdot DE = cd \cdot de$, or $\triangle\, CDE = \triangle\, cde$.

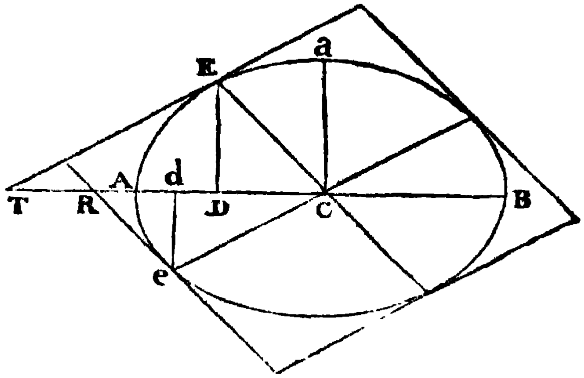
Corol. 3. Alſo $cd^2 = CD \cdot DT$,
and $CD^2 = cd \cdot dR$.

Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dR.

PROPOSITION XIV.

The ſame Figure being conſtructed as in the laſt Propoſition, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the ſame Ratio.

That is, $DA : DT :: DC : DB$,
and $dA : dR :: dC : dB$.



For, by prop. 8, $CD : CA :: CA : CT$,
and by div. $CD : CA :: AD : AT$,
and by comp. $CD : DB :: AD : DT$,
or $DA : DT :: DC : DB$.
In like manner $dA : dR :: dC : dB$. Q.E.D.

COROL. 1. Hence, and from cor. 3 to the laſt prop. we have

$$cd^2 = CD \cdot DT = AD \cdot DB = CA^2 - CD^2,$$
$$CD^2 = cd \cdot dR = Ad \cdot dB = CA^2 - cd^2.$$

Corol. 2. Hence alſo $CA^2 = CD^2 + cd^2$,
and $ca^2 = DE^2 + de^2$.

Corol. 3. Farther, becauſe $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 ; DE^2$,
therefore $CA : ca :: cd ; DE$.
likewiſe $CA : ca :: CD : de$.

PROPOSITION XV.

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point;

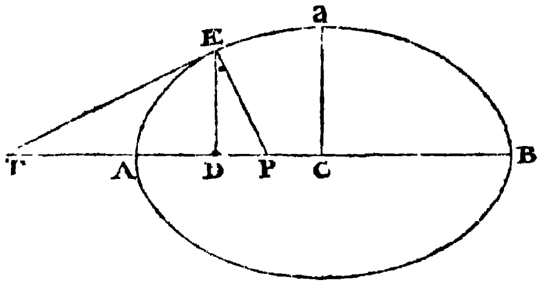
The Diſtance on the Tranſverſe, between the Center and Ordinate, CD:

Will be to the Diſtance PD ::

As the Square of the Tranſverſe Axis:

To the Square of the Conjugate.

That is, $CA^2 : Ca^2 :: DC : DP$.

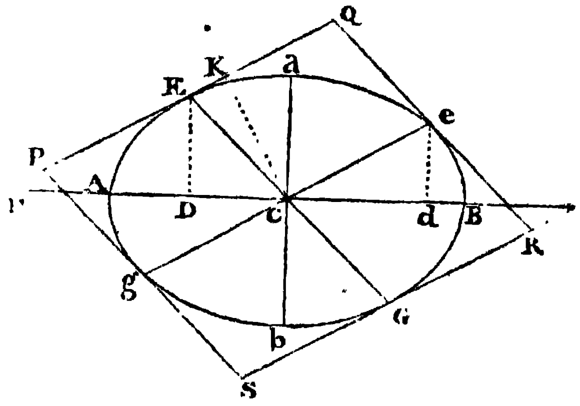


For, by prop. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled △ s, the rect. $TD \cdot DP = DE^2$;
and, by cor. 1 prop. 14, $CD \cdot DT = AD \cdot DB$;
therefore $CA^2 : Ca^2 :: TD \cdot DC : TD \cdot DP$,
$AC^2 : Ca^2 :: DC : DP$. Q.E.D.

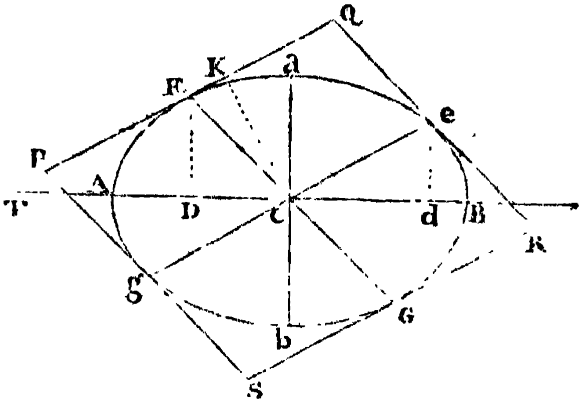
PROPOSITION XVI.

All the Parallelograms circumſcribed about an Ellipſe are equal to one another, and each equal to the Rectangle of the two Axes.

That is, the Parallelogram PQRS = the rectangle AB·ab.



Let EG, eg be two conjugate diameters parallel to the ſides of the parallelogram, and dividing it into four leſſer and equal parallelograms. Alſo draw the ordinates DE, de, and CK perpendicular to PQ.



Then, by prop. 8, CT : CA :: CA : CD ;
and, by cor. 3 prop. 14, ca : de :: CA : CD ;
theref. by equality, CT : CA :: ca : de.
And by ſim. tri. CT : CK :: ce : de,
theref. by equality CK : CA :: ca : ce,
and the rect. CK · ce = rect. CA · ca.
But the rect. CK · ce = the parallelogram CEQe ;
theref. the rect. CA · ca = the parallelogram CEQe ;
and, by doubling, the rect. AB · ab = the paral. PQRS. Q.E.D.

Corol. 1. The rectangles of every pair of conjugate diameters, are to one another reciprocally as the ſines of their included angles. For the areas of their parallelograms, which are all equal among themſelves, are equal to the rectangles of the ſides, or conjugate diameters, multiplied by the ſines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the ſine of their contained angle, is equal to the ſame conſtant quantity. And therefore the rectangle of the diameters is inverſely as the ſine of their contained angle.

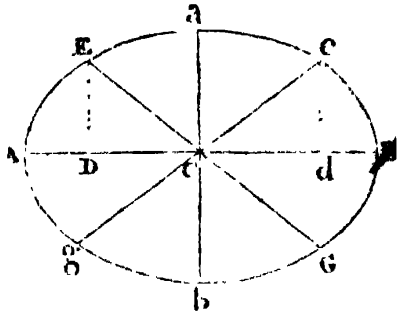
COROL. 2. As it is proved in this propoſition that every circumſcribing parallelogram of an ellipſe is a conſtant quantity, ſo it may hence be ſhewn that each of the ſpaces EAgP, EaeQ, GBeR, Gbgs, between the curve and the tangents, is equal to a conſtant quantity. For, ſince every diameter biſects the ellipſe, the conjugate diameters EG, eg divide the ellipſe into four equal ſectors CEAg, CEae, CGBe, CGbg; but the ſame conjugate diameters divide alſo the whole tangential parallelogram PQRS into four equal parts, or ſmall parallelograms CEPg, CEQe, CGRe, CGSg; and therefore the differences between theſe ſmall parallelograms and the ſectors, which are the ſaid external ſpaces, muſt be all equal among themſelves.

And as the ellipſe and circumſcribing parallelogram both remain conſtant, the difference of their fourth parts will alſo be [illegible] conſtant quantity. That is, the ſaid external parts are each equal to the ſame conſtant quantity.

PROPOSITION XVII.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the ſame conſtant Quantity, namely the Sum of the Squares of the two Axes.

That is, $AE^2 + ab^2 = EG^2 + eg^2$, where EG, eg are any conjugate diameters.



For draw the ordinates ED, ed.

Then, by cor. 2 prop. 14, $CA^2 = CD^2 + cd^2$,

and $ca^2 = DE^2 + de^2$;

therefore the ſum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.

But, by rt. $\angle$ed $\triangle$s, $CE^2 = CD^2 + DE^2$,

and $ce^2 = cd^2 + de^2$;

therefore $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.

conſequently $CA^2 + ca^2 = CE^2 + ce^2$;

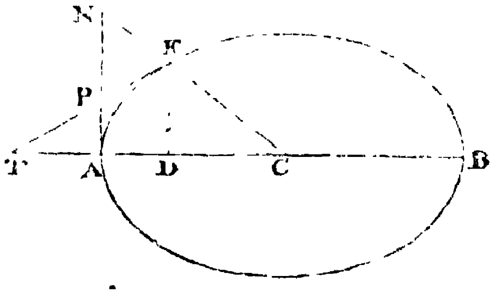
Or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$. Q.E.D.

PRO-

PROPOSITION XVIII.

If there be Two Tangents drawn, the One to the Extremity of the Tranſverſe, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced; the two Tangential Triangles ſo formed, will be equal.

That is, the triangle CET = the triangle CAN.



For, draw the ordinate DE. Then
By ſim. triangles CD : CA :: CE : CN;
but, by prop. 8, CD : CA :: CA : CT;
theref. by equal. CA : CT :: CE : CN.

The two triangles CET, CAN have then the angle C common, and the ſides about that angle reciprocally proportional; therefore thoſe triangles are equal.

Namely the △ CET = △ CAN. Q.E.D.

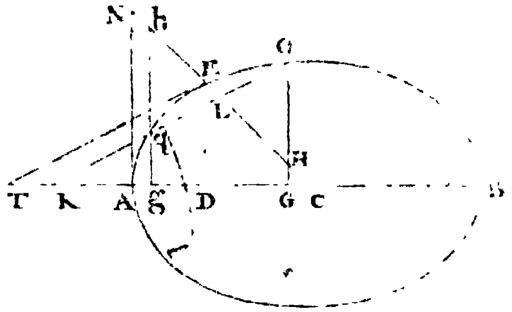
CorOL. 1. From each of the equal tri. CET, CAN,
take the common ſpace CAPE,
and there remains the external △ PAT = △ PNE.

CorOL. 2. Alſo from the equal triangles CET, CAN,
take the common triangle CED,
and there remains the △ TED = trapez. ANED.

PROPOSITION XIX.

The ſame being ſuppoſed as in the laſt Propoſition; then any Lines KQ, GQ, drawn parallel to the two Tangents, ſhall alſo cut off equal Spaces.

That is, the Triangle KQG = Trapez. ANHG.
and the Triangle Kqg = Trapez. ANhg.



For draw the ordinate DE. Then
The three ſim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
theref. by div. the trap. ANED : trap. ANHG :: $CA^2 - CD^2 : CA^2 - CG^2$.
But, by prop. 1, DE^2 : GQ^2 :: $CA^2 - CD^2 : CA^2 - CG^2$.
theref. by equ. trap. ANED : trap. ANHG :: $DE^2 : GQ^2$.
But, by ſim. △s, tri. TED : tri. KQG :: $DE^2 : GQ^2$;
ref. by equal. ANED : TED :: ANHG : KQG.
But, by cor. 2 prop. 18, the trap. ANED = △ TED;
and therefore the trap. ANHG = △ KQG.
In like manner the trap. ANhg = △ Kqg. Q.E.D.

Corol. 1. The three ſpaces ANHG, TEHG, KQG are all equal.

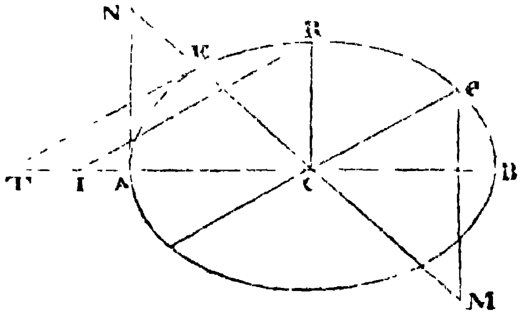
Corol. 2. From the equals ANHG, KQG,
take the equals ANhg, Kqg,
and there remains ghHG = gqQG.

Corol. 3. And from the equals ghHG, gqQG,
take the common ſpace gqLHG,
and there remains the △ LQH = △ Lqh.

Corol. 4. Again from the equals KQG, TEHG,
take the common ſpace KLHG,
and there remains TELK = △ LQH.

Corol. 5. And when, by the lines KQ, GH, moving with a parallel motion, KQ comes into the poſition IR, where CR is the conjugate to CA; then

the triangle KQG becomes the triangle IRC,
and the ſpace ANHG becomes the triangle ANC;
and therefore the △ IRC = △ ANC = △ TEC.



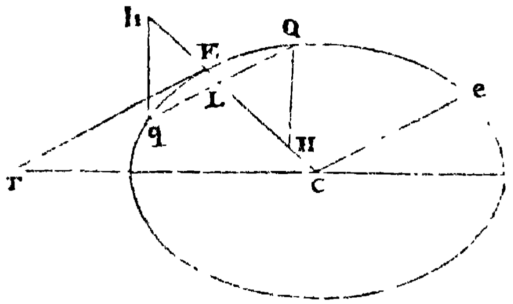
Corol. 6. Alſo when the lines KQ and HQ, by moving with a parallel motion, come into the poſition ce, Me,
the triangle LQH becomes the triangle ceM,
and the ſpace TELK becomes the triangle TEC;
and theref. the △ ceM = △ TEC = △ ANC = △ IRC,

PRO-

PROPOSITION XX.

Any Diameter biſects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.

That is, if Qq be parallel to the Tangent TE, or to Ce, then ſhall LQ = Lq.



For draw QH, qh perpendicular to the tranſverſe.

Then by cor. 3. prop. 18, 19, the △ LQH = △ Lqh;

but theſe triangles are alſo equiangular;

conſequently their like ſides are equal,

and therefore LQ = Lq. Q.E.D.

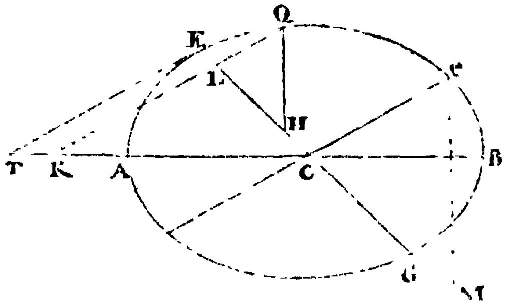
Corol. Any diameter divides the ellipſe into two equal parts.

For, the ordinates on each ſide being equal to each other, and equal in number; all the ordinates, or the area, on one ſide of the diameter, is equal to all the ordinates, or the area, on the other ſide of it.

PROPOSITION XXI.

As the Square of any Diameter :
Is to the Square of its Conjugate : :
So is the Rectangle of any two Abſciſſes :
To the Square of their Ordinate.

That is $CE^2 : ce^2 :: EL \cdot LG$ or $CE^2 - CL^2 : LQ^2$.



For draw the tangent TE, and produce the ordinate QL to the tranſverſe at K. Alſo draw QH, eM perpendicular to the tranſverſe, and meeting EG in H and M.

Then ſimilar triangles being as the ſquares of their like ſides, we ſhall have,

by ſim. triangles, $\triangle\ CET : \triangle\ CLK :: CE^2 : CL^2$;
or, by diviſion, $\triangle\ CET$: trap. $TELK :: CE^2 : CE^2 - CL^2$.
Again, by ſim. tri. $\triangle\ ceM : \triangle\ LQH :: ce^2 : LQ^2$.
But, by cor. 5 prop. 19, the $\triangle\ ceM = \triangle\ CET$,
and, by cor. 4 prop. 19, the $\triangle\ LQH =$ trap. $TELK$;
theref. by equality, $CE^2 : ce^2 :: CE^2 - CL^2 : LQ^2$,
or $CE^2 : ce^2 :: EL \cdot LG : LQ^2$. Q.E.D.

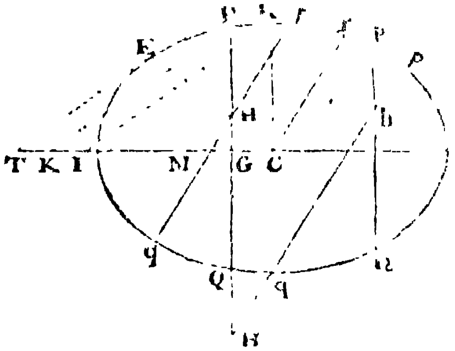
Corol. 1. The ſquares of the ordinates to any diameter, are to one another as the rectangles of their reſpective abſciſſes, or as the difference of the ſquares of the ſemi-diameter and of the diſtance between the ordinate and center. For they are all in the ſame ratio of cE^2 to ce^2.

Corol. 2. The above being the ſame property as that belonging to the two axes, all the other properties before laid down, for the axes, may be underſtood of any two conjugate diameters whatever, uſing only the oblique ordinates of theſe diameters inſtead of the perpendicular ordinates of the axes; namely, all the properties in propoſitions 7, 8, 9, 12, 13, 14, 18 and 19.

PROPOSITION XXII.

If any Two Lines, that any where interſect each other, meet the Curve each in Two Points; then
The Rectangle of the Segments of the one:
Is to the Rectangle of the Segments of the other::
As the Square of the Diam. Parallel to the former:
To the Square of the Diam. Parallel to the latter.

That is, if CR and cr be Parallel to any two Lines PHQ, pHq; then ſhall $CR^2 : cr^2 :: PH \cdot HQ : pH \cdot Hq$.



For draw the diameter CHE, and the tangent TE and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, becauſe ſimilar triangles are as the ſquares of their like ſides, we have,

by ſim. triangles, $CR^2 : GP^2 :: \triangle CRI : \triangle GPK$,
and $CR^2 : GH^2 :: \triangle CRI : \triangle GHM$;
theref. by diviſion, $CR^2 : GP^2 - GH^2 :: CRI : KPHM$.
Again, by ſim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
and by diviſion, $CE^2 : CE^2 - CH^2 :: \triangle CTE : TEHM$.

But, by cor. 5 prop. 19, the $\triangle\, CTE = \triangle\, CIR$,
and by cor. 1 prop. 19, $TEHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CE^2 - CH^2 :: CR^2 : GP^2 - GH^2$ or $PH \cdot HQ$.
In like manner $CE^2 : CE^2 - CH^2 :: cr^2 : pH \cdot Hq$.
Theref. by equal $CR^2 : cr^2 :: PH \cdot HQ : pH \cdot Hq$. Q.E.D.

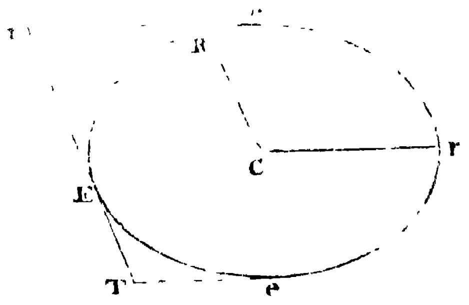
Corol. 1. In like manner, if any other line $p'H'q'$, parallel to cr or to pq, meet PHQ; ſince the rectangles $PH'Q$, $p'H'q'$ are alſo in the ſame ratio of CR^2 to cr^2; theref. the rect. $PHQ : pHq :: PH'Q : p'H'q'$.

Alſo, if another line $P'hQ'$ be drawn parallel to PQ or CR; becauſe the rectangles $P'hQ'$, $p'hq'$ are ſtill in the ſame ratio, therefore, in general,

the rectangle $PHQ : pHq :: P'hQ' : p'hq'$.

That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where interſecting the former.

Corol. 2. And when any of the lines only touch the curve, inſtead of cutting it, the rectangles of ſuch become ſquares, and the general property ſtill attends them.



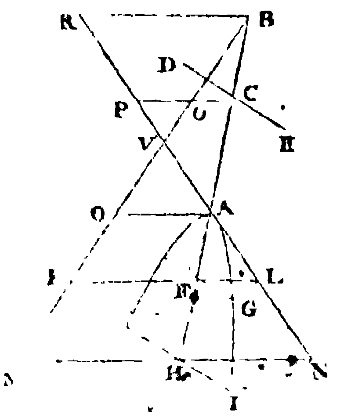
That is, $CR^2 : cr^2 :: TE^2 : Te^2$,
or $CR : cr :: TE : Te$.
and $CR : cr :: tE : te$.

Corol. 3. And hence $TE : Te :: tE : te$.

OF THE HYPERBOLA.

PROPOSITION I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abſciſſes.



Let AVB be a plane paſſing through the vertex and axis of the oppoſite cones; AGIH another ſection of them perpendicular to the plane of the former; AB the axis of the hyperbolic ſections; and FG, HI ordinates perpendicular to it. Then

$\mathrm{FG}^2 : \mathrm{HI}^2 :: \mathrm{AF} \cdot \mathrm{FB} : \mathrm{AH} \cdot \mathrm{HB}$.

For, through the ordinates FG, HI draw the circular ſections KGL, MIN parallel to the baſe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the hyperbola.

Now, by the ſimilar triangles AFL, AHN, and BFK, BHM,

we have $\mathrm{AF} : \mathrm{AH} :: \mathrm{FL} : \mathrm{HN}$,

and $\mathrm{FB} : \mathrm{HB} :: \mathrm{KF} : \mathrm{MH}$;

hence, taking the rectangles of the correſponding terms, we have the rect. $\mathrm{AF} \cdot \mathrm{FB} : \mathrm{AH} \cdot \mathrm{HB} :: \mathrm{KF} \cdot \mathrm{FL} : \mathrm{MH} \cdot \mathrm{HN}$.

But, by the nature of the circle, $\mathrm{KF} \cdot \mathrm{FL} = \mathrm{FG}^2$, and $\mathrm{MH} \cdot \mathrm{HN} = \mathrm{HI}^2$;

Therefore the rect. $\mathrm{AF} \cdot \mathrm{FB} : \mathrm{AH} \cdot \mathrm{HB} :: \mathrm{FG}^2 : \mathrm{HI}^2$. Q.E.D.

CoROL. 1. All the parallel ſections are ſimilar figures, or have their two axes in the ſame proportion; that is, AB : ab :: DE : de.

For, by ſim. triang. AB : ab :: AQ : aq,
and AB : ab :: RB : rb;
Theref. by comp. $AB^2 : ab^2 :: AQ \cdot RB : aq \cdot rb$.
But $AQ \cdot RB = DE^2$, and $aq \cdot rb = de^2$;
Therefore $AB^2 : ab^2 :: DE^2 : de^2$,
or AB : ab :: DE : de.

CoROL. 2. Hence alſo, as the property is the ſame for the ordinates on both ſides of the diameter, it follows, that

1ſt. At equal diſtances from the center, or from the vertices, the ordinates on both ſides are equal, or that the double ordinates are biſected by the axis; and that the whole figure, made up of all the double ordinates, is alſo biſected by the axis.

2d. The two foci are equally diſtant from the center, or from either vertex.

CoROL. 3. When the angle, which the plane of the ſection makes with the baſe of the cone, decreaſes till it become equal to the angle made by the ſide of the cone and the baſe, or till the ſection be parallel to the oppoſite ſide of the baſe; then the axis becomes infinitely long, and the hyperbola degenerates into a parabola; and becauſe then the infinites FB and HB are in a ratio of equality, the general property,

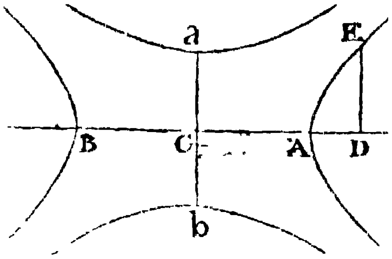
namely $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$,
becomes $AF : AH :: FG^2 : HI^2$,

or, in the parabola, the abſciſſes are to each other, as the ſquares of their ordinates.

PROPOSITION II.

As the Square of the Tranſverſe Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abſciſſes :
To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$



For, by prop. 1. $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
But, if C be the center, then $AC \cdot CB = AC^2$, and ca is the ſemi-conj.
Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 ; ab^2 :: AD \cdot DB : DE^2$. Q.E.D.

Corol. 1. Or, becauſe the rectangle $AD \cdot DB = CA^2 - CD^2$, the ſame property is $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
or $AB^2 : ab^2 :: CD^2 - CA^2 : DE^2$.

Corol. 2. Or, by div. $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$,
that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the tranſverſe,
Is to its parameter,
So is the rectangle of the abſciſſes,
To the ſquare of their ordinate.

COROL. 3. When the axis AB is infinitely long, the curve becomes a parabola, and the infinites AB, DB are then in a ratio of equality; and then the laſt property,

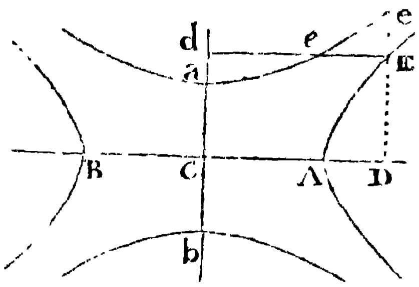
namely $AB : p :: AD \cdot DB : DE^2$;
or $AB \cdot DE : AD \cdot DB :: p : DE$,
becomes $DE : AD :: p : DE$,
or $AD : DE :: DE : p$.

That is, in the parabola, the parameter is a third proportional to any abſciſs and its ordinate.

PROPOSITION III.

As the Square of the Conjugate Axis:
To the Square of the Tranſverſe Axis ::
So is the Sum of the Squares of the Semi-conjugate and Diſtance of the Center from any Ordinate of this Axis:
To the Square of that Ordinate.

That is, $ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.



For draw the ordinate ED to the tranſverſe AB.

Then, by prop. I. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$.

But $CD^2 = dE^2$, and $DE^2 = cd^2$,

therefore $CA^2 : ca^2 :: dE^2 - CA^2 : cd^2$,

or by alternation, $CA^2 : dE^2 - CA^2 :: ca^2 : cd^2$,

and by compoſition, $CA^2 : dE^2 :: ca^2 : ca^2 + cd^2$,

and by alter. & inverſ. $ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.

In like manner $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q.E.D.

Corol. By the laſt prop. $CA^2 : Ca^2 :: CD^2 - CA^2 : DE^2$,
and by this prop. $CA^2 : Ca^2 :: CD^2 + CA^2 : De^2$,
therefore $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner $de^2 : dE^2 :: cd^2 - ca^2 : cd^2 + ca^2$.

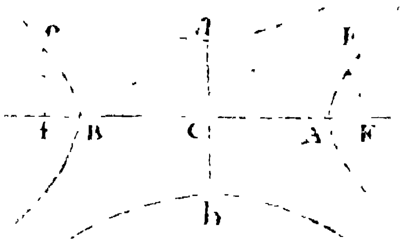
PROPOSITION IV.

The Square of the Diſtance of the Focus from the Center, is equal to the Sum of the Squares of the Semi-axes.

Or, the Square of the Diſtance between the Foci, is equal to the Sum of the Squares of the two Axes.

That is, $CF^2 = CA^2 + Ca^2$,

or $Ff^2 = AB^2 + ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the ſemi-parameter. Then by the nature of the curve $CA^2 : Ca^2 :: CF^2 - CA^2 : FE^2$; and by the def. of the para. $CA^2 : Ca^2 :: Ca^2 : FE^2$;

therefore $Ca^2 = CF^2 - CA^2$;

and by addit. $CF^2 = CA^2 + Ca^2$;

or, by doubling, $Ff^2 = AB^2 + ab^2$. Q.E.D.

COROL. 1. The two ſemi-axes, and the focal diſtance from the center, are the ſides of a right angled triangle CAa; and the diſtance Aa is = CF the focal diſtance.

Fo

For, as above, $CA^2 + ca^2 = CF^2$,
and by right angled △s, $ca^2 + ca^2 = Aa^2$,
therefore $CF = Aa$, and $Ff = Aa + Ba$.

COROL. 2. The conjugate ſemi-axis ca is a mean proportional between AF, FB, or between Af, fB, the diſtances of either focus from the two vertices.

For $ca^2 = CF^2 - CA^2 = CF + CA \cdot CF - CA = AF \cdot FB$.

COROL. 3. The ſame rectangle $AF \cdot FB$ of the focal diſtances from either vertex, is alſo equal to the rectangle $AC \cdot FE$ under the ſemi-tranſverſe and its ſemi-parameter; ſince this laſt is equal to the ſquare of the ſemi-conjugate by the definition of the parameter.

Or $AF : FE :: AC \quad FB$.

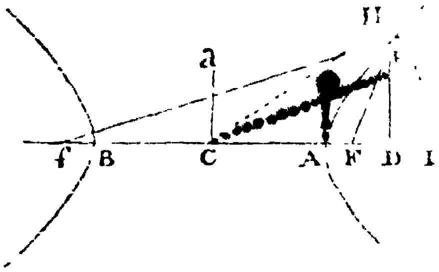
PROPOSITION V.

The Difference between the Semi-tranſverſe and a Line drawn from the Focus to any Point in the Curve, is equal to a Fourth Proportional to the Semi-tranſverſe, the Diſtance from the Center to the Focus, and the Diſtance from the Center to the Ordinate belonging to that Point of the Curve.

That is, $AC - FE = CI$, or $FE = AI$;

and $fE - AC = CI$, or $fE = BI$.

Where $CA : CF :: CD : CI$ the 4th proportional to CA, CF, CD.



For, by right angled triangles, $FE^2 = FD^2 + DE^2$.

Now draw AG parallel and equal to Ca the ſemi-conjugate; and join CG meeting the ordinate DE produced in H.

Then, by prop. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by ſim. △s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
conſequently $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.
Alſo $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
therefore $FE^2 = CF^2 - Ca^2 - 2CF \cdot CD + CD^2 + DH^2$.

But by prop. 4. $CF^2 + Ca^2 = CA^2$
and, by ſuppoſition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$.

But, by ſuppoſition $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by ſim. $\triangle$s, $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore $CI^2 = CD^2 + DH^2 = CH^2$;
conſequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or ſide of this ſquare is $FE = CI - CA = AI$.

In the ſame manner is found $fE = CI + CA = BI$. Q.E.D.

Corol. 1. Hence $CH = CI$ is a 4th proportional to CA, CF, CD.

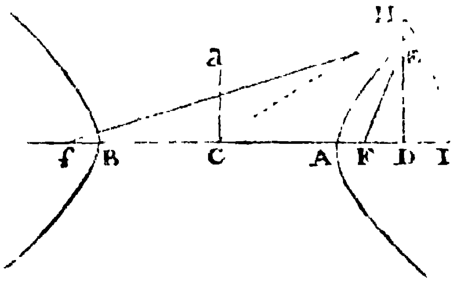
Corol. 2. And $fE + FE = 2CH$ or $2CI$; or FE, CH, fE are in continued arithmetical progreſſion, the common difference being CA the ſemi-tranſverſe.

Corol. 3. From the demonſtration it appears that $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$. Conſequently DH is every where greater than DE; and ſo the aſymptote CGH never meets the curve, though they be ever ſo far produced: but DH and DE approach nearer and nearer to a ratio of equality as they recede farther from the vertex, till at an infinite diſtance they become equal, and the aſymptote is a tangent to the curve at an infinite diſtance from the vertex.

PROPOSITION VI.

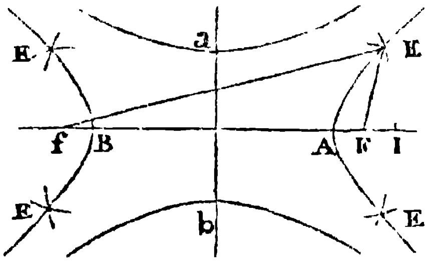
The Difference of two Lines drawn from the Foci, to meet in any Point of the Curve, is equal to the Tranſverſe Axis.

That is, $fE - FE = AB$.



For, by the laſt prop. $FE = CI - CA = AI$,
and, by the ſame, $fE = CI + CA = BI$;
theref. by ſubtraction, $fE - FE = AB$.

Corol. Hence is derived the common method of deſcribing the curve mechanically by points, thus.



In the tranſverſe AB, produced, take the foci F, f, and any point I. Then with the radii AI, BI, and centers F, f, deſcribe arcs interſecting in E, which will be a point in the curve. In like manner, aſſuming other points I, as many other points will be found in the curve.

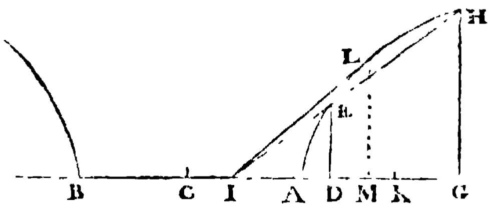
Then with a ſteady hand, draw the curve line through all the points of interſection E.

In the ſame manner are conſtructed the other two hyperbolas, uſing the axis ab inſtead of AB.

PROPOSITION VII.

If from any Point I in the Axis produced, a Line IEH be drawn cutting the curve in Two Points; and from thoſe Two Points be drawn the Perpendicular Ordinates DE, GH; and if K be the Middle of DG, and C the Center or the Middle of AB: Then ſhall CK be to CI as the Rectangle of AD and AG to the Square of AI.

That is, $CK : CI :: AD \cdot AG : AI^2$.



For, by prop. I. $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,

and by ſim. △s, $ID^2 : IG^2 :: DE^2 : GH^2$;

theref. by equal. $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = 2CK - AG$, and $GB = 2CK - AD$,

theref. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$,

and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$.

or $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$,

or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;

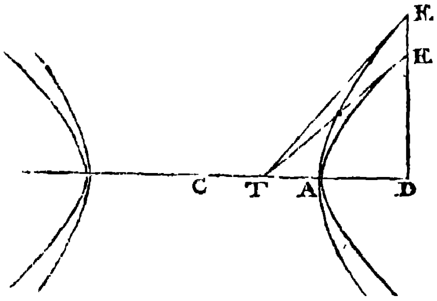
theref. by div. $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$,
and, by div. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or $CK : CI :: AD \cdot AG : AI^2$. Q.E.D.

Corol. When the line IH, by revolving about the point I, comes into the poſition of the tangent IL, and the ordinate LM being drawn, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the property in the propoſition becomes $CM : CI :: AM^2 : AI^2$.

PROPOSITION VIII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Center.

That is, CA is a mean proportion between CD and CT; or CD, CA, CT are continued proportionals.



For, by cor. prop. 7, $CD : CT :: AD^2 : AT^2$,

$$\begin{array}{llll}
\text{that is,} & CD : CT :: & CD - CA : & CA - CT, \\
\text{or} & :: & CD^2 + CA^2 : & CA^2 + CT^2, \\
\text{and} & :: & CD^2 : & CA^2, \\
\text{and} & :: & CA^2 : & CT^2; \\
\text{therefore} & CD : CA :: & CA & CT. \quad \text{Q.E.D.}
\end{array}$$

COROL. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from E, of every hyperbola deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

COROL. 2. Hence a tangent is eaſily drawn to the curve, from any point, either in the curve or without it.

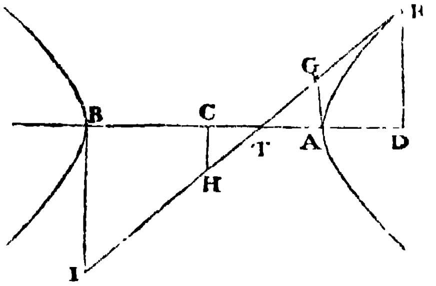
Firſt, if the given point E be in the curve. Draw the ordinate DE of the diameter AC; and in the diameter produced take CT a third proportional to CD, CA. Then join TE for the tangent required.

But if the point T be given any where without the curve. Join CT, in which take CD a third proportional to CT, CA; and draw the ordinate DE. Then join TE as before.

PROPOSITION IX.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four Points, namely the Center, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That is, AG : DE :: CH : BI.



For, by prop. 8, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by ſim. tri. AG : DE :: CH : BI. Q.E.D.

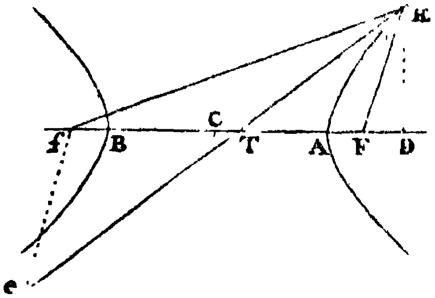
COROL. $\left.\begin{array}{r}\text{Hence TA, TD, TC, TB}\\ \text{and TG, TE, TH, TI}\end{array}\right\}$ are alſo proportionals.

For theſe are as AG, DE, CH, BI, by ſimilar triangles.

PROPOSITION X.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; theſe two Lines will make equal Angles with the Tangent.

That is, the ∠ FET = ∠ fEe.



For draw the ordinate DE, and fe parallel to FE.

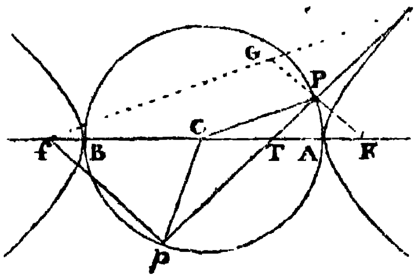
By cor. 1. prop. 5,	CA : CD :: CF : CA + FE,
and by prop. 8,	CA : CD :: CT : CA;
therefore	CT : CF :: CA : CA + FE;
and by add. and ſub.	TF : Tf :: FE : 2CA + FE or fE by prop. 6.
But by ſim. tri.	TF : Tf :: FE : fe;
therefore	fE = fe, and conſeq. ∠ e = ∠ fEe.
But, becauſe	FE is parallel to fe, the ∠ e = ∠ FET;
therefore the	∠ FET = ∠ fEe. Q.E.D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from our propoſition, that rays of light iſſuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reaſon why the points F, f are called *foci*, or burning points.

PROPOSITION XI.

If a Line be drawn from either Focus, Perpendicular to a Tangent to any Point of the Curve; the Diſtance of their Interſection from the Center will be equal to the Semi-tranſverſe Axis.

That is, if FP, fp be perpendicular to the Tangent TPp, then ſhall CP and cp be each equal to CA or CB.



For, through the point of contact E draw FE and fE meeting FP produced in G. Then, the ∠ GEP = ∠ FEP, being each equal to the ∠ fEp, and the angles at P being right, and the ſide PE being common, the two triangles GEP, FEP are equal in all reſpects, and ſo GE = FE, and GP = FP. Therefore, ſince FP = $\frac{1}{2}$ FG, and FC = $\frac{1}{2}$ Ff, and the angle at F common, the ſide CP will be = $\frac{1}{2}$fG or $\frac{1}{2}$AB, that is CP = CA or CB.

And in the ſame manner cp = CA or CB. Q.E.D.

Corol. 1. A circle deſcribed on the tranſverſe axis, as a diameter, will paſs through the points P, p; becauſe all the lines CA, CP, Cp, CB, being equal, will be radii of the circle.

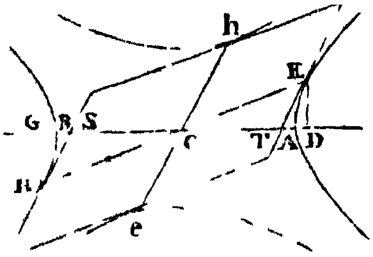
Corol. 2. CP is parallel to fE, and Cp parallel to FE.

Corol. 3. If at the interſections of any tangent, with the circumſcribed circle, perpendiculars to the tangent be drawn, they will meet the tranſverſe axis in the two foci. That is, the perpendiculars PF, pf give the foci F, f.

PROPOSITION XII.

The equal Ordinates, or the Ordinates at equal Diſtances from the Center, on the oppoſite Sides and Ends of an Ellipſe, have their Extremities connected by one Right Line paſſing through the Center, and that Line is biſected by the Center.

That is, if CD = CG, or the ordinate DE = GH; then ſhall CE = CH, and ECH will be a right line.



For, when CD = CG, then alſo is DE = GH by cor. 2. prop. 1. But the ∠D = ∠G, being both right angles; therefore the third ſide CE = CH, and the ∠ DCE = ∠ GCH, and conſequently ECH is a right line.

Corol. 1. And, converſely, if ECH be a right line paſſing through the center; then ſhall it be biſected by the center, or have CE = CH; alſo DE will be = GH, and CD = CG.

Corol. 2. Hence alſo, if two tangents be drawn to the two ends E, H of any diameter EH; they will be parallel to each other, and will cut the axis at equal angles, and at equal diſtances from the center. For, the two CD, CA being equal to the two CG, CB, the third proportionals CT, CS will be equal alſo; then the two ſides CE, CT being equal to the two CH, CS, and the included angle ECT equal to the included angle HCS, all the other correſponding parts are equal: and ſo the ∠T = ∠S, and TE parallel to HS.

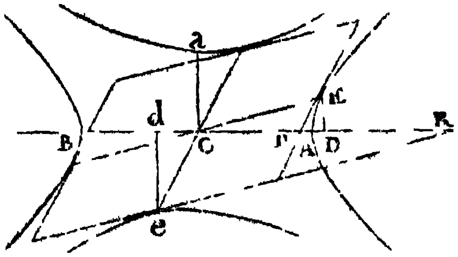
Corol. 3. And hence the four tangents, at the four extremities of any two conjugate diameters, form a parallelogram circumſcribing the ellipſe, and the pairs of oppoſite ſides are each equal to the correſponding parallel conjugate diameters.

For, if the diameter eh be drawn parallel to the tangent TE or HS, it will be the conjugate to EH by the definition; and the tangents to eh will be parallel to each other, and to the diameter EH for the ſame reaſon.

PROPOSITION XIII.

If two Ordinates ED, ed be drawn from the Extremities E, e, of two Conjugate Diameters, and Tangents be drawn to the ſame Extremities, and meeting the Axis produced in T and R;

Then ſhall CD be a mean Proportional between cd, dR, and cd a mean Proportional between CD, DT.



For, by prop. 8, CD : CA :: CA : CT,
and by the ſame cd : CA :: CA : CR;
theref. by equality CD : cd :: CR : CT.
But by ſim. tri. DT : cd :: CT : CR;
theref. by equality CD : cd :: cd : DT.
In like manner cd : CD :: CD : dR. Q.E.D.

COROL. 1. Hence CD : cd :: CR : CT.

COROL. 2. Hence alſo CD : cd :: de : DE.
And the rect. CD·DE = cd·de, or △ CDE = △ cde.

COROL. 3. Alſo $cd^2 = CD \cdot DT$,
and $CD^2 = cd \cdot dR$.

Or cd a mean proportional between CD, DT;
and CD a mean proportional between cd, dR.

PROPOSITION XIV

The ſame Figure being conſtructed as in the laſt Propoſition, each Ordinate will divide the Axis, and the Semi-axis added to the external Part, in the ſame Ratio.

That is, $DA : DT :: DC : DB$,
and $dA : dR :: dc : dB$.

[See the laſt fig.]

For, by prop. 8, $CD : CA :: CA : CT$,
* and by div. $CD : CA :: AD : AT$,
and by comp. $CD : DB :: AD : DT$,
or $DA : DT :: DC : DB$.
In like manner $dA : dR :: dc : dB$. Q.E.D.

Corol. 1. Hence, and from cor. 3 to the laſt prop. we have

$$cd^2 = CD \cdot DT = AD \cdot DB = CD^2 - CA^2,$$
$$CD^2 = cd \cdot dR = Ad \cdot dB = CA^2 - cd^2.$$

Corol. 2. Hence alſo $CA^2 = CD^2 - cd^2$,
and $ca^2 = de^2 - DE^2$.

Corol. 3. Farther, becauſe $CA^2 : ca^2 :: AD \cdot DB$ or $cd^2 : DE^2$,
therefore $CA : ca :: cd : DE$.
likewiſe $CA : ca :: CD : de$.

PROPOSITION XV.

If from any Point in the Curve there be drawn an Ordinate, and a Perpendicular to the Curve, or to the Tangent at that Point;

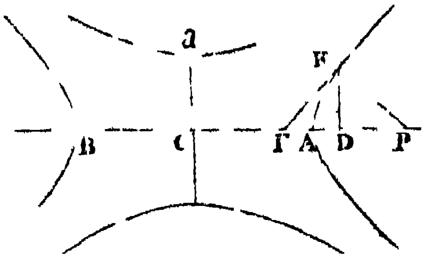
The Diſtance on the Tranſverſe, between the Center and Ordinate, CD:

Will be to the Diſtance PD ::

As the Square of the Tranſverſe Axis:

To the Square of the Conjugate.

That is, $CA^2 : Ca^2 :: DC : DP$.

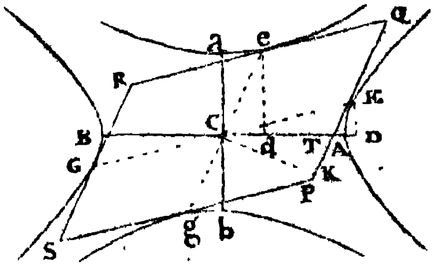


For, by prop. 2, $CA^2 : ca^2 :: AD \cdot DB : DE^2$,
But, by rt. angled △s, the rect. $TD \cdot DP = DE^2$;
and, by cor. 1 prop. 14, $CD \cdot DT = AD \cdot DB$;
therefore $CA^2 : Ca^2 :: TD \cdot DC : TD \cdot DP$,
or $CA^2 : Ca^2 :: DC : DP$. Q.E.D.

PROPOSITION XVI.

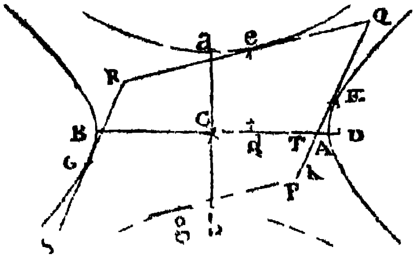
All the Parallelograms inſcribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is, the Parallelogram PQRS = the rectangle AB·ab.



Let EG, eg be two conjugate diameters parallel to the ſides of the parallelogram, and dividing it into four leſſer and equal parallelograms. Alſo draw the ordinates DE, de, and CK perpendicular to PQ.

Then,



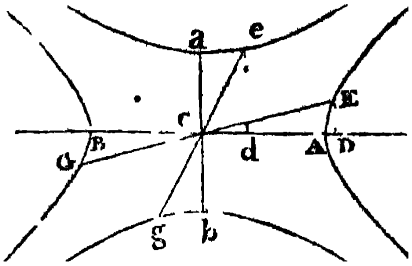
Then, by prop. 8, CT . CA .: CA : CD ;
and, by cor. 3 prop. 14, ca : de :: CA : CD ;
theref. by equality, CT CA . ca : de.
And by ſim. tri. CT . CK .. ce . de,
theref. by equality CK : CA .: ca . ce,
and the rect. CK•ce = rect. CA•ca.
But the rect. CK•ce = the parallelogram CEQe ;
theref. the rect. CA•ca = the parallelogram CEQe ;
and, by doubling, the rect. AB•ab = the paral. PQRS. Q.E.D.

COROL. 1. The rectangles of every pair of conjugate diameters, are to one another reciprocally as the ſines of their included angles. For the areas of their parallelograms, which are all equal among themſelves, are equal to the rectangles of the ſides, or conjugate diameters, multiplied by the ſines of their contained angles, the radius being 1. That is, the rectangle of every two conjugate diameters, drawn into the ſine of their contained angle, is equal to the ſame conſtant quantity. And therefore the rectangle of the diameters is inverſely as the ſine of their contained angle.

PROPOSITION XVII.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the ſame conſtant Quantity, namely the Difference of the Squares of the two Axes.

That is, $AB^2 - ab^2 = EG^2 - eg^2$, where EG, eg are any conjugate diameters.



For draw the ordinates ED, ed.

Then, by cor. 8 prop. 9, $CA^2 = CD^2 - cd^2$,

and $ca^2 = de^2 - DE^2$;

theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.

But, by rt. ∠ed △s, $CE^2 = CD^2 + DE^2$,

and $ce^2 = cd^2 + de^2$;

therefore $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$

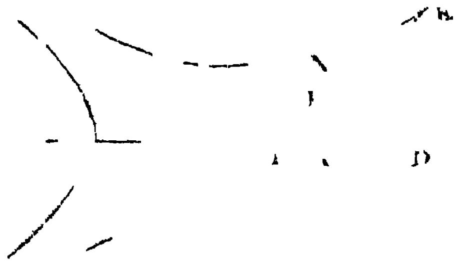
conſequently $CA^2 - ca^2 = CE^2 - ce^2$;

Or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q.E.D.

PROPOSITION XVIII.

If there be Two Tangents drawn, the One to the Extremity of the Tranſverſe, and the other to the Extremity of any other Diameter, each meeting the other's Diameter produced; the two Tangential Triangles ſo formed, will be equal.

That is, the triangle CET = the triangle CAN



For, draw the ordinate DE. Then

By ſim. triangles CD : CA :: CE : CN ;

but, by prop. 8, CD : CA :: CA : CT ;

theref. by equal. CA : CT :: CE : CN.

The two triangles CET, CAN have then the angle common, and the ſides about that angle reciprocally proportional; therefore thoſe triangles are equal.

Namely the △ CET = △ CAN. Q.E.D

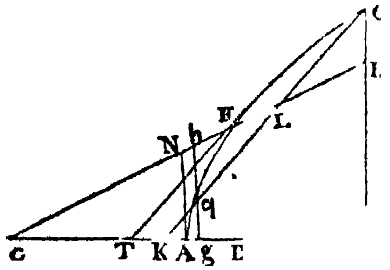
CoROL. 1. Take each of the equal tri. CET, CAN,
from the common ſpace CAPE,
and there remains the external △ PAT = △ PNE.

CoROL. 2. Alſo take the equal triangles CET, CAN,
from the common triangle CED,
and there remains the △ TED = trapez. ANED.

PROPOSITION XIX.

The ſame being ſuppoſed as in the laſt Propoſition; then any Lines KQ, GQ, drawn parallel to the two Tangents, ſhall alſo cut off equal Spaces.

That is, the Triangle KQG = Trapez. ANHG.
and the Triangle Kqg = Trapez. ANhg.



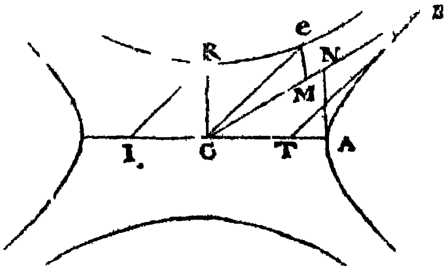
For draw the ordinate DE. Then
The three ſim. triangles CAN, CDE, CGH,
are to each other as CA^2, CD^2, CG^2;
theref. by div. the trap. ANED : trap. ANHG :: $CD^2 - CA^2 : CG - CA^2$.
But, by prop. 1, $DE^2 : GQ^2 :: CD^2 - CA^2 : CG^2 - CA^2$.
theref. by equ. trap. ANED : trap. ANHG :: $DE^2 : GQ^2$.
But, by ſim. △s, tri. TED : tri. KQG :: $DE^2 : GQ^2$;
theref. by equal. ANED : TED :: ANHG : KQG.
But, by cor. 2 prop. 18, the trap. ANED = △ TED;
and therefore the trap. ANHG = △ KQG.
In like manner the trap. ANhg = △ Kqg. Q.E.D.

Corol. 1. The three ſpaces ANHG, TEHG, KQG are all equal.

CoROL. 2. From the equals ANHG, KQG,
take the equals ANhg, Kqg,
and there remains ghHG = gqQG.

CoROL. 3. And from the equals ghHG, gqQG,
take the common ſpace gqLHG,
and there remains the △LQH = △Lqh.

CoROL. 4. Again from the equals KQG, TEHG,
take the common ſpace KLHG,
and there remains TELK = △LQH.



CoROL. 5. And when, by the lines KQ, GH, moving with a parallel motion, KQ comes into the poſition IR, where CR is the conjugate to CA; then

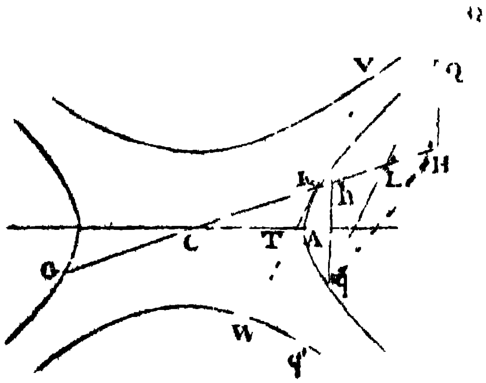
the triangle KQG becomes the triangle IRC,
and the ſpace ANHG becomes the triangle ANC;
and therefore the △IRC = △ANC = △TEC.

CoROL. 6. Alſo when the lines KQ and HQ, by moving with a parallel motion, come into the poſition ce, Me,
the triangle LQH becomes the triangle ceM,
and the ſpace TELK becomes the triangle TEC;
and theref. the △ceM = △TEC = △ANC = △IRC,

PROPOSITION XX.

Any Diameter biſects all its Double Ordinates, or the Lines drawn Parallel to the Tangent at its Vertex, or to its Conjugate Diameter.

That is, if Qq be parallel to the Tangent TE, or to ce, then ſhall LQ = Lq.



For draw QH, qh perpendicular to the tranſverſe.
Then by cor. 3 prop. 19, the △ LQH = △ Lqh;
but theſe triangles are alſo equiangular;
conſequently their like ſides are equal,
and therefore LQ = Lq. Q.E.D.

Corol. 1. Any diameter divides the ellipſe into two equal parts.

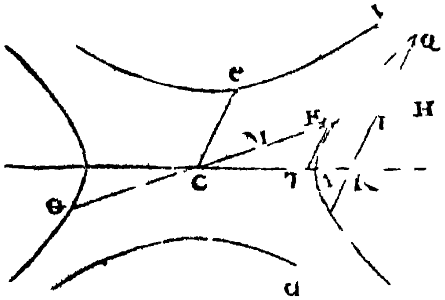
For, the ordinates on each ſide being equal to each other, and equal in number; all the ordinates, or the area, on one ſide of the diameter, is equal to all the ordinates, or the area, on the other ſide of it.

Corol. 2. In like manner, if the ordinate be produced to the conjugate hyperbolas at Q′, q′, it may be proved that LQ′ = Lq′. Or if the tangent TE be produced, then EV = EW. And the diameter GCEH biſects all lines drawn parallel to TE or Qq, and limited either by one hyperbola, or by its two conjugate hyperbolas.

PROPOSITION XXI.

As the Square of any Diameter :
Is to the Square of its Conjugate : :
So is the Rectangle of any two Abſciſſes :
To the Square of their Ordinate.

That is $CE^2 : ce^2 :: EL \cdot LG$ or $CL^2 - CE^2 : LQ^2$



For draw the tangent TE, and produce the ordinate QL to the tranſverſe at K. Alſo draw QH, eM perpendicular to the tranſverſe, and meeting EG in H and M.

Then ſimilar triangles being as the ſquares of their like ſides, we ſhall have,

by ſim. triangles, $\triangle CET : \triangle CLK :: CE^2 : CL^2$;
or, by diviſion, $\triangle CET :$ trap. $TELK :: CE^2 : CL^2 - CE^2$.
Again, by ſim. tri. $\triangle CeM : \triangle LQH :: ce^2 : LQ^2$.
But, by cor. 5 prop. 19, the $\triangle CeM = \triangle CET$,
and, by cor. 4 prop. 19, the $\triangle LQH =$ trap. $TELK$;
theref. by equality, $CE^2 : ce^2 :: CL^2 - CE^2 : LQ^2$,
or $CE^2 : ce^2 :: EL \cdot LG : LQ^2$. Q.E.D.

Corol. 1. The ſquares of the ordinates to any diameter, are to one another as the rectangles of their reſpective abſciſſes, or as the difference of the ſquares of the ſemi-diameter and of the diſtance between the ordinate and center. For they are all in the ſame ratio of CE^2 to ce^2.

Corol. 2. The above being the ſame property as that belonging to the two axes, all the other properties before laid down, for the axes, may be underſtood of any two conjugate diameters whatever, uſing only the oblique ordinates of theſe diameters inſtead of the perpendicular ordinates of the axes; namely, all the properties in propoſitions 7, 8, 9, 12, 13, 14, 18 and 19.

Corol. 3. Likewiſe, when the ordinates are continued to the conjugate hyperbolas at Q', q', the ſame properties ſtill obtain, ſubſtituting only the ſum for the difference of the ſquares of CE and CL,

That is, $CE^2 : ce^2 :: CL^2 + CE^2 : CQ'^2$.
And ſo $LQ^2 : LQ'^2 :: CL^2 - CE^2 : CL^2 + CE^2$.

Corol. 4. When, by the motion of LQ parallel to itſelf, that line coincides with EV, the laſt corollary becomes $CE^2 : ce^2 :: 2CE^2 : EV^2$, [*]
or $ce^2 : EV^2 :: 1 : 2$,
or $ce : EV :: 1 : \sqrt{2}$,
or as the ſide of a ſquare to its diagonal.

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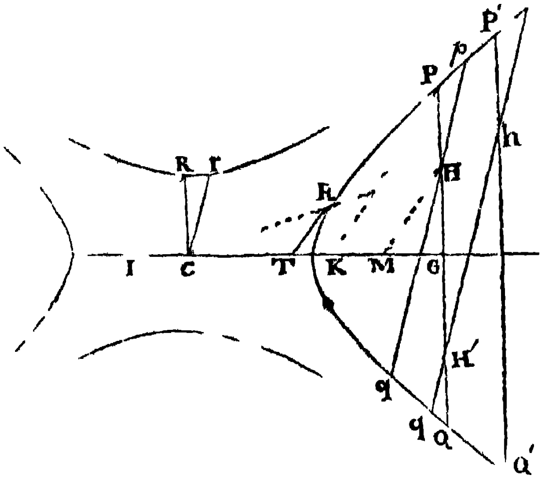
[*] That is, in all conjugate hyperbolas; and all their diameters, any diameter is to its parallel tangent, in the conſtant ratio of the ſide of a ſquare to its diagonal.

PROPOSITION XXII.

If any Two Lines, that any where interſect each other, meet the Curve each in Two Points; then
The Rectangle of the Segments of the one:
Is to the Rectangle of the Segments of the other::
As the Square of the Diam. Parallel to the former:
To the Square of the Diam. Parallel to the latter.

That is, if CR and cr be Parallel to any two Lines PHQ, pHq; then ſhall $CR^2 : cr^2 :: PH \cdot HQ : pH \cdot Hq$.



For draw the diameter CHE, and the tangent TE and its parallels PK, RI, MH, meeting the conjugate of the diameter CR in the points T, K, I, M. Then, becauſe ſimilar triangles are as the ſquares of their like ſides, we

by ſim. triangles, $CR^2 : GP^2 :: \triangle CRI : \triangle GPK$,
and $CR^2 : GH^2 :: \triangle CRI : \triangle GHM$;
theref. by diviſion, $CR^2 . GP^2 - GH^2 :: CRI : KPHM$.
Again, by ſim. tri. $CE^2 : CH^2 :: \triangle CTE : \triangle CMH$;
and by diviſion, $CE^2 : CH^2 - CE^2 :: \triangle CTE : TEHM$.

But, by cor. 5 prop. 19, the $\triangle CTE = \triangle CIR$,
and by cor. 1 prop. 19, $TEHG = KPHG$, or $TEHM = KPHM$;
theref. by equ. $CE^2 : CH^2 - CE^2 :: CR^2 : GP^2 - GH^2$ or $PH \cdot HQ$.
In like manner $CE^2 : CH^2 - CE^2 :: cr^2 : pH \cdot Hq$.
Theref. by equal $CR^2 : cr^2 :: PH \cdot HQ : pH \cdot Hq$. Q.E.D.

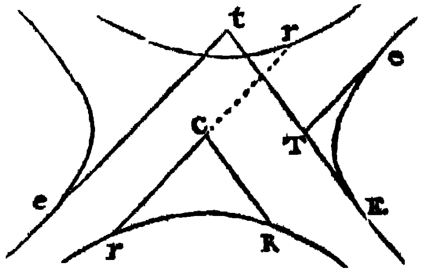
Corol. 1. In like manner, if any other line $p'H'q'$, parallel to cr or to pq, meet PHQ; ſince the rectangles $PH'Q$, $p'H'q'$ are alſo in the ſame ratio of CR^2 to cr^2; theref. the rect. $PHQ : pHq :: PH'Q : p'H'q'$.

Alſo, if another line $P'hQ'$ be drawn parallel to PQ or CR; becauſe the rectangles $P'hQ'$, $p'hq'$ are ſtill in the ſame ratio, therefore, in general,

the rectangle $PHQ : pHq :: P'hQ' : p'hq'$.

That is, the rectangles of the parts of two parallel lines, are to one another, as the rectangles of the parts of two other parallel lines, any where interſecting the former.

Corol. 2. And when any of the lines only touch the curve, inſtead of cutting it, the rectangles of ſuch become ſquares, and the general property ſtill attends them.



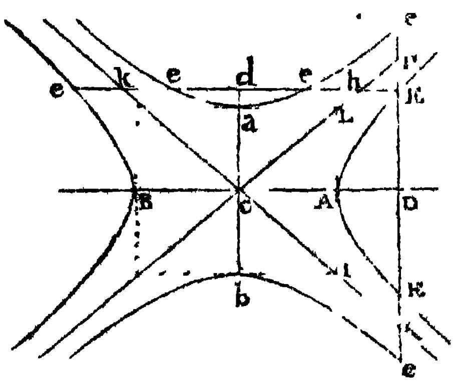
That is, $CR^2 : cr^2 :: TE^2 : Te^2$,
or $CR : cr :: TE : Te$.
and $CR : cr :: tE : te$.

Corol. 3. And hence $TE : Te :: tE : te$.

PROPOSITION XXIII.

If a Line be drawn through any Point of the Curves, Parallel to either of the Axes, and terminated at the Aſymptotes; the Rectangle of its Segments, meaſured from that Point, will be equal to the Square of the Semi-axis to which it is parallel.

That is, the Rect. HEK or HeK $= ca^2$,
and the Rect. hEk or hek $= CA^2$.



For draw AL parallel to ca, and aL to CA. Then
by the parallels, $CA^2 : ca^2$ or $AL^2 :: CD^2 : DH^2$;
and, by prop. 2, $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$;
theref. by ſubtr. $CA^2 : ca^2 :: CA^2 : DH^2 - DE^2$ or HEK.
But the antecedents CA^2, CA^2 are equal,
theref. the conſequents ca^2, HEK muſt alſo be equal.

In like manner we have, again

by the parallels, $CA^2 : Ca^2$ or $AL^2 :: CD^2 : DH^2$;

and, by prop. 3, $CA^2 : Ca^2 :: CD^2 + CA^2 : De^2$;

theref. by ſubtr. $CA^2 : Ca^2 :: CA^2 : De^2 - DH^2$ or HeK.

But the antecedents CA^2, CA^2 are the ſame,

theref. the conſequents Ca^2, HeK muſt be equal.

In like manner, by changing the axes, is hEk or $hek = CA^2$. Q.E.D.

Corol. 1. Becauſe the rect. HEK = the rect. HeK.

therefore $EH : eH :: eK : EK$.

And conſequently HE is always greater than He.

Corol. 2. The rectangle hEK = the rect. HEk.

For, by ſim. tri. $Eh : EH :: Ek : EK$.

SCHOLIUM.

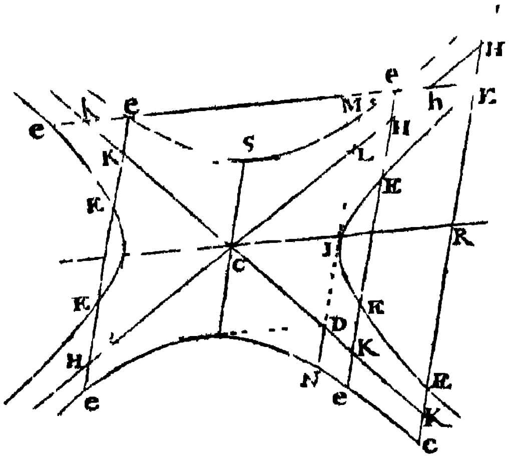
It is evident that this propoſition is general for any line oblique to the axis alſo, namely, that the rectangle of the ſegments of any line, cut by the curve, and terminated by the aſymptotes, is equal to the ſquare of the ſemi-diameter to which the line is parallel. Since the demonſtration is drawn from properties that are common to all diameters.

PRO-

PROPOSITION XXIV.

All the Rectangles are equal which are made of the Segments of any Parallel Lines cut by the Curve, and limited by the Aſymptotes.

That is, the Rect. HEK = Rect. HeK.
and the Rect. hEk = hek.



For each of the rectangles HEK or HeK is equal to the ſquare of the parallel ſemi-diameter CS; and each of the rectangles hEk or hek is equal to the ſquare of the parallel ſemi-diameter CI. And therefore the rectangles of the ſegments of all parallel lines are equal to one another.

Q.E.D.

COROL. 1. The rectangle HEK being conſtantly the ſame, whether the point E is taken on the one ſide or the other of the point of contact I of the tangent parallel to HK, it follows that the parts HE, KE, of any line HK, are equal.

And becauſe the rectangle HeK is conſtant, whether the point e is taken in the one or the other of the oppoſite hyperbolas, it follows that the parts He, Ke are alſo equal.

COROL. 2. And when HK comes into the poſition of the tangent DIL, the laſt corollary becomes $IL = ID$, and $IM = IN$, and $LM = DN$.

Hence alſo the diameter CIR biſects all the parallels to DL which are terminated by the aſymptote, namely $RH = RK$.

COROL. 3. From the propoſition, and the laſt corollary, it follows that the conſtant rectangle HEK or EHE is $= IL^2$. And the equal conſtant rect. HeK or eHe $= MLN$ or $IM^2 - IL^2$.

COROL. 4. And hence $IL =$ the parallel ſemi-diameter CS.

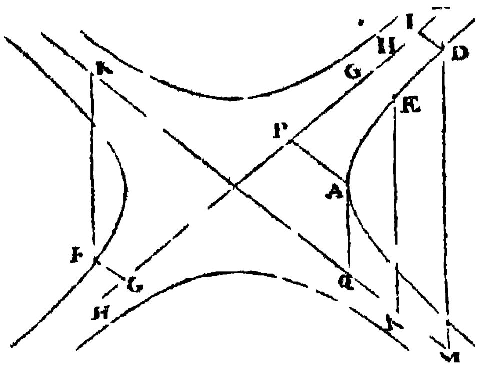
For the rect. EHE $= IL^2$,
and the equal rect. eHe $= IM^2 - IL^2$,
theref. $IL^2 = IM^2 - IL^2$, or $IM^2 = 2IL^2$;
but, by cor. 4 prop. 21, $IM^2 = 2CS^2$,
and therefore $IL = CS$.

And ſo the aſymptotes paſſes through the oppoſite angles of all the inſcribed parallelograms.

PROPOSITION XXV.

The Rectangle of any two Lines drawn from any Point in the Curve, Parallel to two given Lines, and Limited by the Aſymptotes, is a Conſtant Quantity.

That is, if AP, EG, DI be Parallels,
as alſo AQ, EK, DM Parallels,
then ſhall the rect. PAQ = rect. GEK = rect IDM.

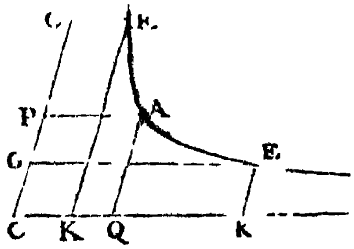


For, produce KE, MD to the other aſymptote at H, L.
Then, by the parallels, HE : GE :: LD : ID;
but, EK : EK :: DM : DM;
theref. the rectangle HEK : GEK :: LDM : IDM.
But, by the laſt prop. the rect. HEK = LDM;
and therefore the rect. GEK = IDM = PAQ. Q.E.D

PROPOSITION XXVI.

All the Parallelograms are equal which are formed between the Aſymptotes and Curve, by Lines Parallel to the Aſymptotes.

That is, the paral. CGEK = CPAQ.



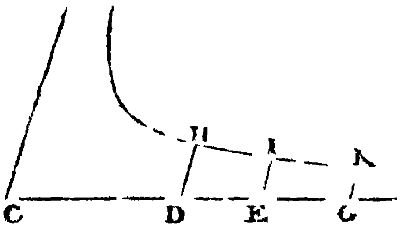
For, by prop. 25, the rect. GEK or CGE = rect. PAQ or CPA.

But the parallelograms CGEK, CPAQ being equiangular, are as the rectangles CGE, CPA.

And therefore the parallelogram GK = PQ.

That is, all the inſcribed parallelograms are equal to one another. Q.E.D.

CoROL. 1. Becauſe the rectangle GEK or CGE is conſtant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the aſymptote continually approaches towards the curve, but never meets it: for GE decreaſes continually as CG increaſes; and it is always of *ſome* magnitude, except when CG is ſuppoſed to be infinitely great, for then GE is infinitely ſmall, or nothing. So that the aſymptote CG may be conſidered as a tangent to the curve at a point infinitely diſtant from C.



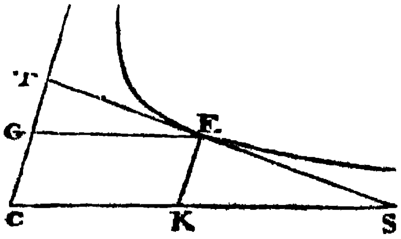
CoROL. 2. If the abſciſſes CD, CE, CG, &c, taken on the one aſymptote, be in geometrical progreſſion increaſing; then ſhall the ordinates DH, EI, GK, &c, parallel to the other aſymptote, be a decreaſing geometrical progreſſion, having the ſame ratio. For, all the rectangles CDH, CEI, CGK, &c, being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abſciſſes CD, CF, CG, &c, which are geometricals. And the reciprocals of geometricals are alſo geometricals, and in the ſame ratio.

CoROL. 3. But if the abſciſſes CD, CE, CG, &c, be arithmeticals, or in arithmetical progreſſion, then ſhall the ordinates DH, EI, GK, &c, be harmonicals, or in harmonical progreſſion. For the reciprocals of arithmeticals, are harmonicals.

PROPOSITION XXVII.

Every Inſcribed Triangle, formed by Any Tangent and the two Intercepted Parts of the Aſymptotes, is equal to a Conſtant Quantity; namely Double the Inſcribed Parallelogram.

That is, the triangle CTS = 2 paral. GK.

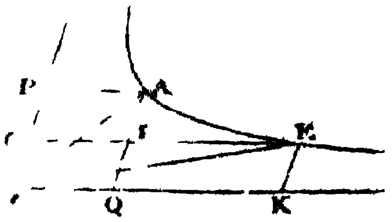


For, ſince the tangent TS is biſected by the point of contact E, and EK is parallel to TC, and GE to CK; therefore CK, KS, GE are all equal, as are alſo CG, GT, KE. Conſequently the triangle GTE = the triangle KES, and each equal to half the conſtant inſcribed parallelogram GK. And therefore the whole triangle CTS, which is compoſed of the two ſmaller triangles and the parallelogram, is equal to double the conſtant inſcribed parallelogram GK. Q.E.D.

PROPOSITION XXVIII.

The three following Spaces, between the Aſymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Center, and each of the two Quadrilaterals, contained by the ſaid Arc, and two Lines drawn from its Extremities parallel to one Aſymptote, and the intercepted Part of the other Aſymptote.

That is, the ſector CAE = PAEG = QAEK,
all ſtanding upon the ſame arc AE.



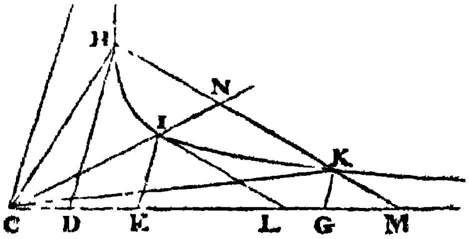
For, by prop. 26, CPAQ = CGEK;
ſubtract the common ſpace CGIQ,
ſo ſhall the paral. PI = the par. IK;
to each add the trilineal IAE,
then is the quadr. PAEG = QAEK.
Again, from the quadrilateral CAEK
take the equal triangles CAQ, CEK,
and there remains the ſector CAE = QAEK.
Therefore CAE = QAEK = PAEG. Q.E.D.

PRO-

PROPOSITION XXIX.

If from the Point of Contact of any Tangent, and the two Interſections of the Curve with a Line parallel to the Tangent, three parallel Lines be drawn in any Direction, and terminated by either Aſymptote; thoſe three Lines ſhall be in continued Proportion.

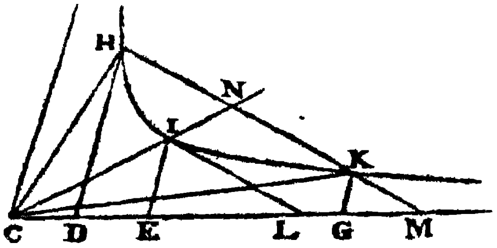
That is, if HKM and the tangent IL be parallel, then are the parallels DH, EI, GK in continued proportion.



For, by the parallels, $EI : IL :: DH : HM$;
and, by the ſame, $EI : IL :: GK : KM$;
theref. by compoſ. $EI^2 : IL^2 :: DH \cdot GK : HMK$;
but, by prop. 24, the rect. $HMK = IL^2$;
and theref. the rect. $DH \cdot GK = EI^2$,
or $DH : EI :: EI : GK$. Q.E.D.

PROPOSITION XXX.

Draw the ſemi-diameters CH, CIN, CK;
Then ſhall the ſector CHI = the ſector CIK.



For, becauſe HK and all its parallels are biſected by CIN, therefore the triangle CNH = tri. CNK,
and the ſegment INH = ſeg. INK;
conſequently the ſector CIH = ſec. CIK.

COROL. 2. If the geometricals DH, EI, GK be parallel to the other aſymptote, the ſpaces DHIE, EIKG will be equal; for they are equal to the equal ſectors CHI, CIK.

So that by taking any geometricals CD, CE, CG, &c, and drawing DH, EI, GK, &c, parallel to the other aſymptote, as alſo the radii CH, CI, CK.

then the ſectors CHI, CIK, &c,
or the ſpaces DHIE, EIKG, &c,
will be all equal among themſelves.
Or the ſectors CHI, CHK, &c,
or the ſpaces DHIE, DHKG, &c,
will be in arithmetical progreſſion.

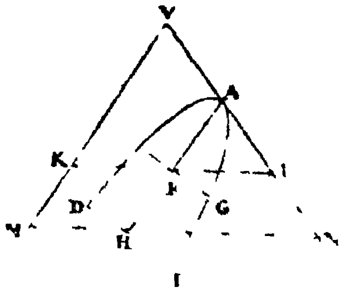
And therefore theſe ſectors, or ſpaces, will be analogous to the logarithms of the lines or baſes CD, CE, CG, &c; namely CHI or DHIE the log. of the ratio of
CD to CE, or of CE to CG, &c; or of EI to DH, or of GK to EI, &c;
and CHK or DHKG the log. of the ratio of
CD to CG, &c, or of GK to DH, &c.

OF THE PARABOLA.

PROPOSITION I.

The Abſciſſes are Proportional to the Squares of their Ordinates.

Let AVM be a ſection through the axis of the cone, and AGIH a parabolic ſection by a plane perpendicular to the former, and parallel to the ſide VN of the cone; alſo let AFH be the common interſection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.



Then I ſay that $AF : AH :: FG^2 : HI^2$.

For, through the ordinates FG, HI draw the circular ſections KGL, MIN parallel to the baſe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

Then by ſimilar triangles $AF : AH :: FL : HN$;
but, becauſe of the parallels, $KF = MH$;
therefore $AF : AH :: KF \cdot FL : MH \cdot HN$.
But, by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;
Therefore $AF : AH :: FG^2 : HI^2$. Q.E.D.

Corol. 1. Hence the third proportional $\frac{EG^2}{AE}$ or $\frac{HI^2}{AH}$ is a conſtant quantity, and is equal to the parameter of the axis by cor. 5 prop. 1 of the ellipſe.

Or AE : EG :: EG : P the parameter.

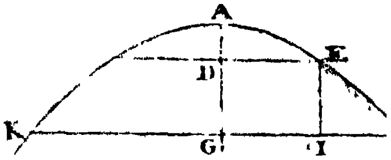
Or the rectangle $P \cdot AE = EG^2$

Corol. 2. Hence alſo as the property is the ſame for the ordinates on both ſides of the diameter, it follows that the ordinates DH, HI, on both ſides of the axis are equal, or that the double ordinates are biſected by the axis; and that the whole figure, made up of all the double ordinates, is alſo biſected by the axis.

PROPOSITION II.

As the Parameter of the Axis :
Is to the Sum of any Two Ordinates : :
So is the Difference of thoſe Ordinates :
To the Difference of their Abſciſſes :

That is, $P : GH + DE :: GH - DE : DG$,
Or, $P : KI :: IH : EI$.



For, by cor. 1 prop. 1, $P \cdot AG = GH^2$,
and $P \cdot AD = DE^2$;
theref. by ſubtrac. $P \cdot DG = GH^2 - DE^2$.

But $GH^2 - DE^2 = \overline{GH + DE} \cdot \overline{GH - DE} = KI \cdot IH$,
theref. the rectangle $P \cdot DG = KI \cdot IH$,
or $P : KI :: IH : DG$ or EI. Q.E.D.

Corol. Hence becauſe $P \cdot EI = KI \cdot IH$,
and, by cor. 1 prop. 1, $P \cdot AG = GH^2$,
therefore $AG : EI :: GH^2 : KI \cdot IH$.

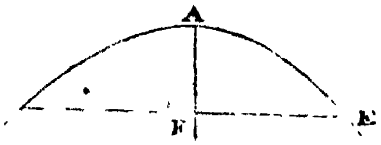
And ſo any diameter EI is as the rectangle of the ſegments KI, IH of the double ordinate KH.

PRO-

PROPOSITION III.

The Diſtance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is, $AF = \frac{1}{2}FE = \frac{1}{4}P$.



For, the general property is $AF : FE :: FE : P$.

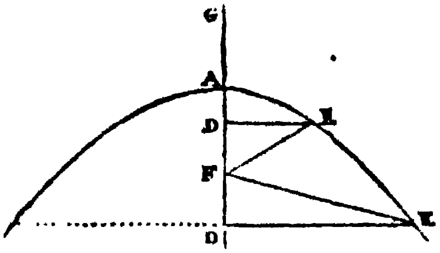
But, by the definition, $FE = \frac{1}{2}P$;

therefore alſo $AF = \frac{1}{2}FE = \frac{1}{4}P$. Q.E.D.

PROPOSITION IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Diſtance and the Abſciſs of the Ordinate to that Point.

That is, $FE = FA + AD = GD$, taking $AG = AF$.



For ſince $FD = AD \backsim AF$,

theref. by ſquaring, $FD^2 = AF^2 - 2AF \cdot AD + AD^2$,

But, by cor. 1 prop. 1, $DE^2 = P \cdot AD = 4AF \cdot AD$;

theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$,

But, by rt. angled $\triangle$s, $FD^2 + DE^2 = FE^2$;

therefore $FE^2 = AF^2 + 2AF \cdot AD + AD^2$,

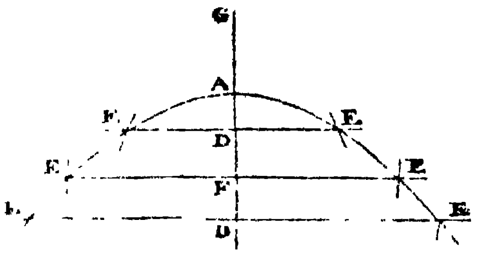
and the root or ſide is $FE = AF + AD$,

or $FE = GD$, by taking $AG = AF$. Q.E.D.

CoROL.

CОROL. 1. The difference FE − FE is always equal to DD the difference of the abſciſſes.

COROL. 2. Hence alſo the curve is eaſily deſcribed



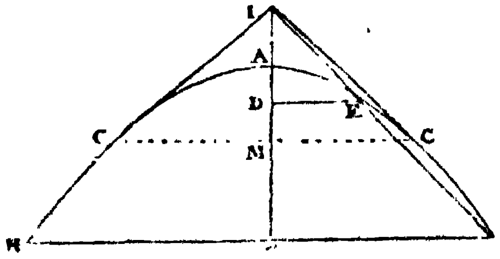
by points. Namely, in the axis produced take AG = AF the focal diſtance, and draw a number of lines EE perpendicular to the axis AD; then with the diſtances GD, GD, GD, &c, as radii, and the centre F, draw arcs croſſing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

PROPOSITION V.

If a Line be drawn from any Point in the Axis produced, to cut the Curve in two Points; then ſhall the external Part of the Axis be a Mean Proportional between the two Abſciſſes of the Ordinates to the two Points of Interſection.

That is, AI a mean proportional between AD, AG,

or, AD : AI :: AI : AG.



For, by prop. I, $AD : AG :: DE^2 : GH^2$;

and, by ſim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$; *

theref. by equal. $AD : AG :: ID^2 : IG^2$;

and, by diviſion, $AD : DG :: ID^2 : IG^2 - ID^2$ or $DG \cdot \overline{ID + IG}$

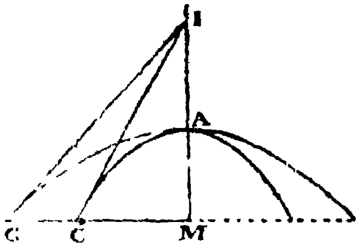
or $AD : ID :: ID : ID + IG$;

and by diviſion $AD : AI :: ID : IG$ *,

and again by div. $AD : AI :: AI : AG$. Q.E.D.

COROL. I. * Hence we have $AD : AI :: DE : GH$.

Corol. 2. If the line be ſuppoſed to revolve about the point I; then as it recedes farther from the axis, the points E and H approach towards each other, the point E deſcending, and the point H aſcending, till at laſt they meet in the point C when the line becomes a tangent to the curve at C. And then the points D and G meet in the point M, and the ordinates DE, GH in the ordinate CM. Conſequently AD, AG, becoming each equal to AM, their mean proportional AI will be equal to the abſciſs AM. That is, the external part of the axis, cut off by a tangent, is equal to the abſciſs of the ordinate to the point of contact.



Corol. 3. Hence the tangents to all parabolas, which have the ſame abſciſſes, meet the axis produced in the ſame point. For if the abſciſs AM be the ſame in all, the external axis AI, which is equal to it, will be the ſame alſo.

Corol. 4. And hence alſo a tangent is eaſily drawn to the curve.

For if the point of contact C be given; draw the ordinate CM, and produce MA till AI be = AM; then join IC the tangent.

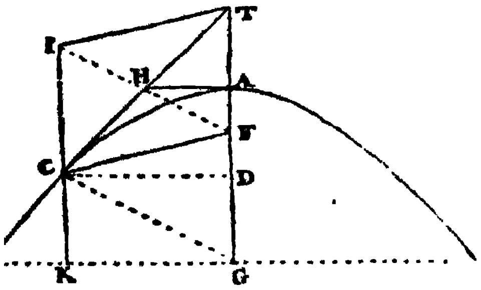
Or if the point I be given; take AM = AI, and draw the ordinate MC, which will give the point of contact C, to which draw IC the tangent.

PRO-

PROPOSITION VI.

If a Tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Diſtance of the Focus from the Interſection of the Tangent and Axis.

That is, FC = FT.



For draw the ordinate DC to the point of contact.
Then, by cor. 1 prop. 5, AT = AD ;
therefore FT = AF + AD.
But, by prop. 4, FC = AF + AD ;
therefore by equality, FC = FT. Q.E.D.

CoROL. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C ; then ſhall FG = FC = FT.

For draw FH perpendicular to TC, which will alſo biſect TC, becauſe FT = FC ; and therefore, by the nature of the parallels, FH alſo biſects TG in F. And conſequently FG = FT = FC.
So that F is the center of a circle paſſing through T, C, G.

CoROL. 2. The ſubnormal DG is every where equal to the conſtant quantity 2FA, or FT, or $\frac{1}{2}$ P the ſemi-parameter.

For draw the tangent AH parallel to DC, making the triangle FHA ſimilar to GCD.

Then DC = 2AH, becauſe DT = 2DA:
conſequently DG = 2FA = $\frac{1}{2}$ P.

CoROL. 3. The tangent at the vertex AH, is a mean proportional between AF and AD.

For becauſe FHT is a right angle,
therefore AH is a mean between AF, AT,
or between AF, AD,
becauſe AD = AT.
Likewiſe FH is a mean between FA, FT,
or between FA, FC.

CoROL. 4. The tangent TC makes equal angles with FC and the axis FT, or with the line ICK drawn parallel to the axis.

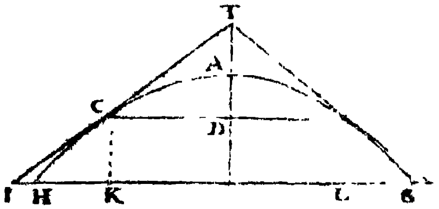
For becauſe FT = FC,
therefore the ∠ FCT = ∠ FTC = its altern. ∠ ICT.
Alſo the angle GCF = the angle GCK.

CoROL. 5. And becauſe the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light falling upon the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

PROPOSITION VII.

If an Ordinate be drawn to the Point of Contact of any Tangent, and another Ordinate produced to cut the Tangent; It will be as the Difference of the Ordinates :
Is to the Difference added to the external Part ::
So is Double the firſt Ordinate :
To the Sum of the Ordinates.

That is, KH : KI :: KL : KG.



For, by cor. 1 prop. 1, P : DC :: DC : DA,
and P : 2DC :: DC : DT or 2DA.
But, by ſim. triangles, KI : KC :: DC : DT;
therefore by equality, P : 2DC :: KI : KC,
or, P : KI :: KL : KC.
Again, by prop. 2, P : KH :: KG : KC;
therefore by equality, KH : KI :: KL : KG. Q.E.D.

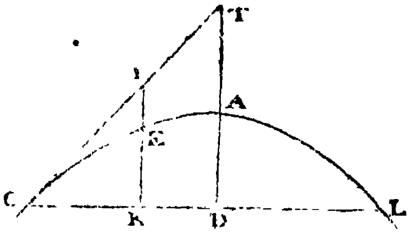
COROL. 1. Hence, by compoſition and diviſion,
we have KH : KI :: GK : GI,
and HI : HK :: HK : KL,
alſo IH : IK :: IK : IG; that is,
IK is a mean proportional between IG and IH.

COROL. 2. And from this laſt property we can eaſily draw a tangent to the curve from any given point I. Namely, draw IHG perpendicular to the axis, and take IK a mean proportional between IH, IG; then draw KC parallel to the axis, and C will be the point of contact, through which and the given point I the tangent IC is to be drawn.

PROPOSITION VIII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and alſo any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then ſhall the Curve divide that Line in the ſame Ratio, as the Line divides the Double Ordinate.

That is, IE : EK :: CK : KL.

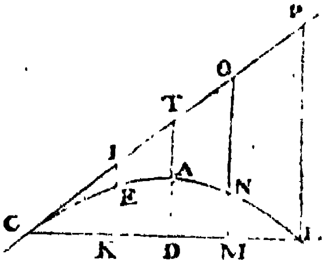


For, by ſim. triangles, CK : KI :: CD : DT or 2DA;
but, by the def. the param. P : CL :: CD : 2DA;
therefore by equality, P : CK : CL : KI.
But, by prop. 2, P : CK :: KL : KE;
therefore by equality, CL : KL :: KI : KE;
and, by diviſion, CK : KL :: IE : EK. Q.E.D.

PROPOSITION IX.

The ſame being ſuppoſed as in Prop. VIII: then ſhall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2.
and IE, TA, ON, PL, &c,
are as CI^2, CT^2, CO^2, CP^2, &c,
or as CK^2, CD^2, CM^2, CL^2, &c.



For, by prop. 8, $IE : EK :: CK : KL$,
or, by equality, $IE : EK :: CK^2 : CK \cdot KL$.
But, by cor. prop. 2, EK is as the rect. $CK \cdot KL$,
and therefore IE is as CK^2, or as CI^2. Q.E.D.

Corol. 1. As this property is common to every poſition of the tangent, if the lines IE, TA, ON, &c, be appended to the points I, T, O, &c, and movable about them, and of ſuch lengths as that their extremities E, A, N, &c, be in the curve of a parabola in ſome one poſition of the tangent; then making the tangent revolve about the point C, the extremities E, A, N, &c, will always form the curve of ſome parabola, in every poſition of the tangent.

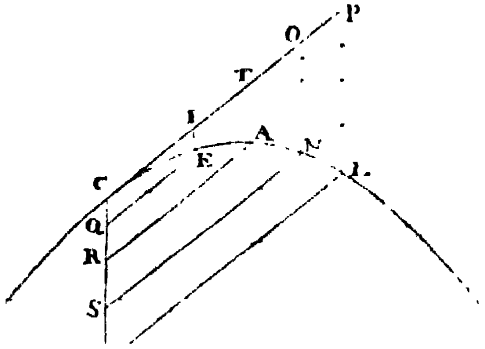
Corol. 2. The parameter of the axis is alſo a third proportional to IE and CK.

For, by this prop. EK : KL :: IE : CK;
and, by prop. 2, EK : KL :: CK : P;
therefore by equality, IE : CK :: CK : P.

PROPOSITION X.

The Abſciſſes of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c,
are as QE^2, RA^2, SN^2, &c.
Or $CQ : CR :: QE^2 : RA^2$, &c.



For, draw the tangent CT, and the externals EI, AT, NO, &c, parallel to the axis, or to the diameter CS.

Then, becauſe the ordinates QE, RA, SN, &c, are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c are parallelograms, whoſe oppoſite ſides are equal,

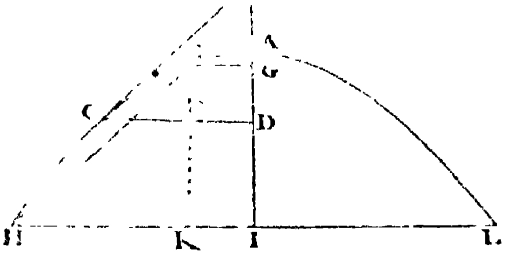
namely IE, TA, ON, &c,
are equal to CQ, CR, CS, &c.
Therefore by prop. 9, CQ, CR, CS, &c,
are as CI^2, CT^2, CO^2, &c,
or as their equals QE^2, RA^2, SN^2, &c. Q.E.D.

COROL. Here, like as in prop. 2, the difference of the abſciſſes; are as the difference of the ſquares of their ordinates, or as the rectangles under the ſum and difference of the ordinates, the rectangle under the ſum and difference of the ordinates being equal to the rectangle under the difference of the abſciſſes and the parameter of that diameter, or a third proportional to any abſciſs and its ordinate.

PROPOSITION XI.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Interſections, and a third to the Point of Contact, theſe three Ordinates will be in Arithmetical Progreſſion, or the Sum of the Extremes will be equal to Double the Mean.

That is EG + HI = 2CD.



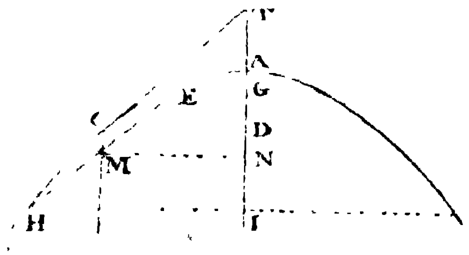
For draw EK parallel to the axis, and produce HI to L.

Then, by ſim. triangles, EK : HK :: TD or 2AD : CD ;
but, by prop. 2, EK : HK :: KL : P the param.
theref. by equality, 2AD : KL :: CD : P.
But, by the defin. 2AD : 2CD :: CD : P ;
theref. the 2d terms are equal, KL = 2CD,
that is EG + HI = 2CD. Q.E.D.

PROPOSITION XII.

Any Diameter biſects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is, ME = MH.



For to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by prop. XI, 2MN or 2CD = EG + HI,
therefore M is the middle of EH.

And for the ſame reaſon all its parallels are biſected. Q.E.D.

SCHOLIUM.

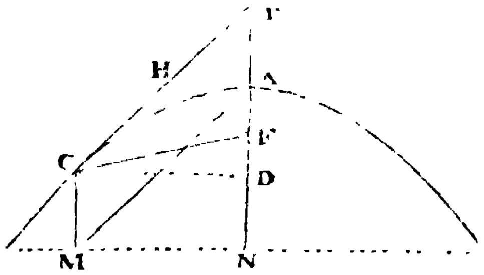
Hence as the abſciſſes of any diameter and their ordinates have the ſame relations as thoſe of the axis, namely, that the ordinates are biſected by the diameter, and their ſquares proportional to the abſciſſes; ſo all the other properties of the axis and its ordinates and abſciſſes, before demonſtrated, will likewiſe hold good for any diameter and its ordinates and abſciſſes. And alſo thoſe of the parameters, underſtanding the parameter of any diameter, as a third proportional to any abſciſs and its ordinate. Some of the moſt material of which are demonſtrated in the following propoſitions.

PRO-

PROPOSITION XIII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, $4\text{FC} = p$ the param. of the diam. CM.



For draw the ordinate MA parallel to the tangent CT; as alſo CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abſciſſes AD, CM or AT being equal by cor. 2 to prop. 5, the parameters will be as the ſquares of the ordinates CD, MA or CT, by the definition;

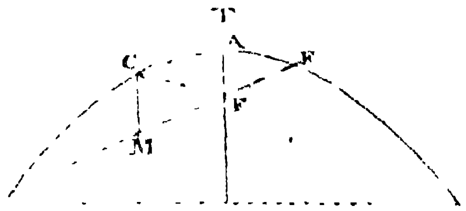
that is, $\text{P} : \text{p} :: \text{CD}^2 : \text{CT}^2$,
But, by ſim. tri. $\text{FH} : \text{FT} :: \text{CD} : \text{CT}$;
therefore $\text{P} : \text{p} :: \text{FH}^2 : \text{FT}^2$.
But, by cor. 3 prop. 6, $\text{FH}^2 = \text{FA} \cdot \text{FT}$;
therefore $\text{P} : \text{p} :: \text{FA} \cdot \text{FT} : \text{FT}^2$,
or, by equal. $\text{P} : \text{p} :: \text{FA} : \text{FT}$ or FC.
But, by prop. 3, $\text{P} = 4\text{FA}$,
and therefore $\text{p} = 4\text{FT}$ or 4FC. Q.E.D.

CoROL. Hence the parameter p of the diameter CM is equal to $4\text{FA} + 4\text{AD}$, or to $\text{P} + 4\text{AD}$, that is, the parameter of the axis added to 4AD.

PROPOSITION XIV.

If an Ordinate to any Diameter paſs through the Focus, it will be equal to Half its Parameter; and its Abſciſs equal to One Fourth of the ſame Parameter.

That is, CM $= \frac{1}{4}p$, and ME $= \frac{1}{2}p$.



For join FC, and draw the tangent CT.

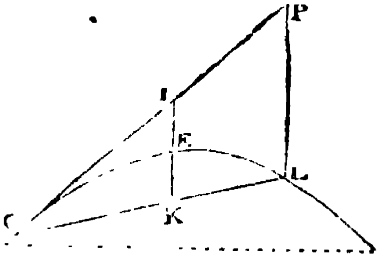
By the parallels, CM $=$ FT;
and, by prop. 6, FC $=$ FT;
alſo, by the laſt prop. FC $= \frac{1}{4}$p;
therefore CM $= \frac{1}{4}$p.
Again, by the defin. CM or $\frac{1}{4}$p : ME :: ME : p,
and conſequently ME $= \frac{1}{2}$p $=$ 2CM.

Corol. Hence, of any diameter, the double ordinate which paſſeth through the focus, is equal to the parameter, or to quadruple its abſciſs.

PROPOSITION XV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in ſome other Point, as alſo another Line parallel to the Axis, and limited by the Firſt Line and the Tangent: then ſhall the Curve divide this Second Line in the ſame Ration, as the Second Line divides the Firſt Line.

That is, IE : EK :: CK : KL.



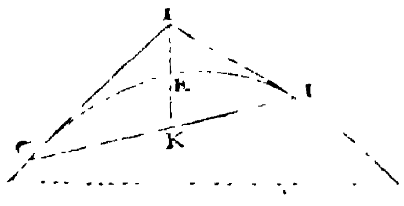
For draw LP parallel to IK or to the axis.

Then, by prop. 9, IE PL $CI^2 : CP^2$,
or by ſim. tri. IE PL $CK^2 : CL^2$.
Alſo, by ſim. tri. IK PL CK : CL,
or IK PL $CK^2 : CK \cdot CL$;
therefore by equality, IE IK $CK \cdot CL : CL^2$,
or IE IK CK : CL;
and, by diviſion, IE EK CK : KL. Q.E.D.

PROPOSITION XVI.

If a Tangent cut any Diameter produced, and if an Ordinate to that Diameter be drawn from the Point of Contact; then the Diſtance in the Diameter produced, between the Vertex and the Interſection of the Tangent, will be equal to the Abſciſs of that Ordinate.

That is, IE = EK.



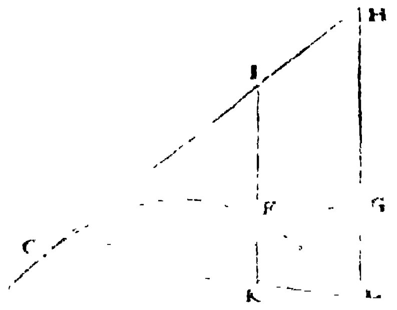
For, by the laſt prop. IE : EK :: CK : KL.
But, by prop. 12, CK = KL,
and therefore IE = EK. Q.E.D.

Corol. 1. The two tangents CI, LI, at the extremities of any double ordinate CL, meet in the ſame point of the diameter of that double ordinate produced. And the diameter drawn through the interſection of two tangents, biſects the line connecting the points of contact.

Corol. 2. Hence we have another method of drawing a tangent from any given point I without the curve. Namely, from I draw the diameter IK, in which take EK = EI, and through K draw CL parallel to the tangent at E; then C and L are the points to which the tangents muſt be drawn from I.

PROPOSITION XVII.

If from any Point of the Curve there be drawn a Tangent, and alſo Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Interſection E and L, meeting thoſe Two Right Lines in two other Points G and K: Then will the Line KG joining theſe laſt Two Points be parallel to the Tangent.



For, by prop. 15, CK : KL :: EI : EK;
and by comp. CK : CL :: EI : KI :: GH : LH by parallels.
But, by ſim. tri. CK : CL :: KI : LH;
theref. by equal. KI : LH :: GH : LH;
conſequently KI = GH,
and therefore KG is parallel and equal to IH. Q.E.D.

OTHERWISE.

By prop. 15, EI : EK :: CK : KL :: CE : EG by the parallels. So that the two triangles CEI, KEG have their angles at E equal, and the ſides about thoſe angles proportional; they are therefore ſimilar; conſequently the angle at C is equal to the angle at G, and KG parallel to CH. Q.E.D.

COROL. 1. Hence we have KI = GH.

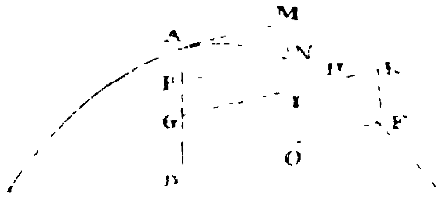
COROL. 2. Alſo KI or GH is a mean proportional between EI and LH.

For, by the parallels, EI : KI :: GH or KI : LH.

PROPOSITION XVIII.

If a Line be drawn from the Vertex of any Diameter to cut the Curve in ſome other Point, and an Ordinate of that Diameter be drawn to that Point, as alſo another Ordinate any where cutting the Line, both produced if neceſſary: The Three will be continual Proportionals, namely, the two Ordinates and the Part of the Latter limited by the ſaid Line drawn from the Vertex.

That is, DE, GH, GI are continual Proportionals,
Or DE : GH :: GH : GI.



For, by prop. 10, $DE^2 : GH^2 :: AD : AG$;
and, by ſim. tri. $DE : GI :: AD : AG$;
theref. by equality $DE : GI :: DE^2 : GH^2$,
that is, of the three DE, GH, GI, 1ſt : 3d :: $1ſt^2 : 2d^2$;
therefore 1ſt : 2d :: 2d : 3d,
that is $DE : GH :: GH : GI$. Q.E.D.

COROL. 1. Or their equals GK, GH, GI are proportionals; where EK is parallel to the diameter AD.

COROL. 2. Hence we have DE : AG :: *p* : GI, where p is
the parameter, or AG : GI :: DE : p.
For, by the defin. AG : GH :: GH : p.

COROL. 3. Hence alſo the three MN, MI, MO are proportionals, where MO is parallel to the diameter, and AM parallel to the ordinates.

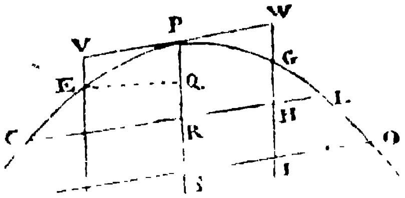
For, by prop. 10,	MN,	MI,	MO,
or their equals	AP,	AG,	AD,
are as the ſquares of	PN,	GH,	DE,
or of their equals	GI,	GH,	GK,

which are proportionals by cor. 1.

PROPOSITION XIX.

If a Diameter cut any Parallel Lines terminated by the Curve; the Segments of the Diameter will be as the Rectangle of the Segments of thoſe Lines.

That is, EK : EM :: CK·KL : NM·MO.
Or, EK is as the rectangle CK·KL.



For draw the diameter PS to which the parallels CL, NO are ordinates, and the ordinate EQ parallel to them.

Then CK is the difference, and KL the ſum of the ordinates EQ, CR; alſo NM the difference, and MO the ſum of the ordinates EQ, NS. And the differences of the abſciſſes, are QR, QS, or EK, EM.

Then by cor. prop. 10, QR : QS :: CK·KL : NM·MO,
that is EK : EM :: CK·KL : NM·MO.

CoROL. 1. The rect. CK·KL = rect. EK and the param. of PS.

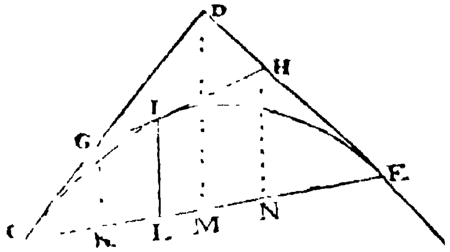
For the rect. CK·KL = rect. QR and the param. of PS.

CoROL.

PROPOSITION XXI.

If there be Three Tangents interſecting each other; their Segments will be in the ſame Proportion.

That is, GI : IH :: CG : GD :: DH : HE.



For through the points G, I, D, H, draw the diameters GK, IL, DM, HN; as alſo the lines CI, EI, which are double ordinates to the diameters GK, HN, by cor. 1 prop. 16; therefore the diameters GK, DM, HN,
biſect the lines CL, CE, LE;
hence KM = CM − CK = $\frac{1}{2}$ CE − $\frac{1}{2}$ CL = $\frac{1}{2}$ LE = LN or NE,
and MN = ME − NE = $\frac{1}{2}$ CE − $\frac{1}{2}$ LE = $\frac{1}{2}$ CL = CK or KL.
But, by parallels, GI : IH :: KL : LN,
and CG : GD :: CK : KM,
alſo DH : HE :: MN : NE.
But the 3d terms KL, CK, MN are all equal;
as alſo the 4th terms LN, KM, NE.

Therefore the firſt and ſecond terms, in all the lines, are proportional, namely GI : IH :: CG : GD :: DH : HE. Q.E.D.

PRAC-

PRACTICAL EXERCISES

IN

MENSURATION.

QUEST. I. WHAT difference is there between a floor 28 feet long by 20 broad, and two others each of half the dimenſions; and what do all three come to at 45s per ſquare, or 100 ſquare feet?

Anſ. dif. 280 ſq. feet. Amount 18 guineas.

QU. 2. An elm plank is 14 feet 3 inches long, and I would have juſt a ſquare yard ſlit off it; at what diſtance from the edge muſt the line be ſtruck? Anſ. $7\frac{99}{171}$ inches.

QU. 3. A ceiling contains 114 yards 6 feet of plaſtering, and the room 28 feet broad; what was the length of it? Anſ. $36\frac{6}{7}$ feet.

QU. 4. A common joiſt is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a ſcantling juſt as big again, that ſhall be 3 inches thick; what will the other dimenſion be?

Anſ. $11\frac{2}{3}$ inches.

QU. 5. A wooden trough coſt me 3s 6d painting within, at 6d per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Anſ. $27\frac{1}{4}$ inches.

Qu. 6. If my court yard be 47 feet 9 inches ſquare, and I have laid a foot-path with Purbeck ſtone, of 4 feet wide, along one ſide of it; what will paving the reſt with flints come to at 6d per ſquare yard? Anſ. £5 16 0½.

Qu. 7. A ladder, 40 feet long, may be ſo planted, that it ſhall reach a window 33 feet from the ground on one ſide of the ſtreet; and, by only turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high on the other ſide: what is the breadth of the ſtreet? Anſ. 56 feet 7¾ inches.

Qu. 8. The paving of a triangular court, at 18d per foot, came to 100l; the longeſt of the three ſides was 88 feet; required the ſum of the other two equal ſides.

Anſ. 106·85 feet.

Qu. 9. There are two columns in the ruins of Perſepolis left ſtanding upright; the one is 64 feet above the plain, and the other 50: in a ſtraight line between theſe ſtands an ancient ſmall ſtatue, the head of which is 97 feet from the ſummit of the higher, and 86 feet from the top of the lower column, the baſe of which meaſures juſt 76 feet to the center of the figure's baſe. Required the diſtance between the tops of the two columns.

Anſ. 157 feet nearly.

Qu. 10. The perambulator, or ſurveying wheel, is ſo contrived, as to turn juſt twice in the length of a pole, or 16½ feet; required the diameter. Anſ. 2·626 feet.

Qu. 11. In turning a one-horſe chaiſe within a ring of a certain diameter, it was obſerved that the outer wheel made two turns while the inner made but one: the wheels were both 4 feet high; and, ſuppoſing them fixed at the ſtatutable diſtance of 5 feet aſunder on the axletree, what was the circumference of the track deſcribed by the outer wheel? Anſ. 63 feet nearly.

Qu. 12. What is the ſide of that equilateral triangle whoſe area coſt as much paving at 8d a foot, as the palliſading the three ſides did at a guinea a yard?

Anſ. 72·746 feet.

Qu. 13. In the trapezium ABCD are given AB = 13, BC = 31$\frac{1}{5}$, CD = 24, and DA = 18, alſo B a right angle; required the area. Anſ. 410·122.

Qu. 14. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb per ſquare foot: what will it come to at 18s per cwt? Anſ. £ 22 19 10$\frac{1}{4}$.

Qu. 15. Having a rectangular marble ſlab, 58 inches by 27, I would have a ſquare foot cut off parallel to the ſhorter edge; I would then have the like quantity divided from the remainder parallel to the longer ſide; and this alternately repeated, till there ſhall not be the quantity of a foot left: what will be the dimenſions of the remaining piece? Anſ. 20·7 inches by 6·086.

Qu. 16. Given two ſides of an obtuſe-angled triangle, which are 20 and 40 poles; required the third ſide that the triangle may contain juſt an acre of land..

Anſ. 58·876 or 23·699.

Qu. 17. The end wall of a houſe is 24 feet 6 inches in breadth, and 40 feet to the eaves; $\frac{1}{3}$ of which is 2 bricks thick, $\frac{1}{3}$ more is 1$\frac{1}{2}$ brick thick, and the reſt 1 brick thick. Now the triangular gable riſes 38 courſes of bricks, 4 of which uſually make a foot in depth, and this is but 4$\frac{1}{2}$ inches, or half a brick thick: what will this piece of work come to at 5l 10s per ſtatute rod?

Anſ. £ 20 11 7$\frac{1}{2}$.

Qu. 18. How

Qu. 18. How many bricks will it take to build a wall 10 feet high, and 500 feet long, and a brick and half thick, reckoning the brick 10 inches long, and 4 courſes to the foot in height? Anſ. 36000.

Qu. 19. How many bricks will build a ſquare pyramid, of 100 feet on each ſide at the baſe, and alſo 100 feet perpendicular height: the dimenſions of a brick being ſuppoſed 10 inches long, 5 inches broad, and 3 inches thick.
Anſ. 3840000.

Qu. 20. If from a right-angled triangle, whoſe baſe is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular cutting off a triangle whoſe area is 24 ſquare feet; required the ſides of this triangle.
Anſ. 6, 8, and 10.

Qu. 21. The ellipſe in Groſvenor-ſquare meaſures 840 links acroſs the longeſt way, and 612 the ſhorteſt, within the rails: now the walls being 14 inches thick, what ground do they incloſe, and what do they ſtand upon?

$$\text{Anſ.}\begin{cases}\text{incloſe 4ac or 6p.}\\ \text{ſtand on }1760\tfrac{1}{2}\text{ ſq feet.}\end{cases}$$

Qu. 22. If a round pillar, 7 inches over, have 4 feet of ſtone in it; of what diameter is the column, of equal length, that contains 10 times as much?
Anſ. 22·136 inches.

Qu. 23. A circular fiſh-pond is to be made in a garden, that ſhall take up juſt half an acre; what muſt be the length of the cord that ſtrikes the circle?
Anſ. 27$\frac{3}{4}$ yards.

Qu. 24. When a roof is of a true pitch, the rafters are $\frac{3}{4}$ of the breadth of the building: now ſuppoſing the eaves-boards to project 10 inches on a ſide, what will the new ripping a houſe coſt, that meaſures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15s per ſquare? Anſ. £ 8 15 9½.

Qu. 25. A cable which is 3 feet long, and 9 inches in compaſs, weighs 22 lb; what will a fathom of that cable weigh, which meaſures a foot about? Anſ. 78½ lb.

Qu. 26. My plumber has put 28 lb per ſquare foot into a ciſtern 74 inches and twice the thickneſs of the lead long, 26 inches broad, and 40 deep; he has alſo put three ſtays acroſs it within, 16 inches deep, of the ſame ſtrength, and reckons 22s per cwt, for work and materials. I, being a maſon, have paved him a workſhop, 22 feet 10 inches broad, with Purbeck ſtone, at 7d per foot; and upon the balance I find there is 3s 6d due to him; what was the length of the workſhop? Anſ. 32 f 0¾ inch.

Qu. 27. The diſtance of the centers of two circles, whoſe diameters are each 50, being given equal to 30; what is the area of the ſpace incloſed by their circumferences? Anſ. 559·119.

Qu. 28. If 20 feet of iron railing weigh half a ton when the bars are an inch and quarter ſquare, what will 50 feet come to at 3½d per lb, the bars being but ⅞ of an inch ſquare? Anſ. £20 0 2.

Qu. 29. The area of an equilateral triangle, whoſe baſe falls on the diameter, and its vertex in the middle of the arc of a ſemicircle, is equal to 100: what is the diameter of the ſemicircle? Anſ. 26·32148.

Qu. 30. It is required to find the thickneſs of the lead in a pipe of an inch and quarter bore, which weighs 14 lb per yard in length; the cubic foot of lead weighing 11325 ounces. Anſ. ·20737 inches.

Qu. 31. Suppoſe the expence of paving a ſemicircular plot, at 2s 4d per foot, come to 10l, what is the diameter of it? Anſ. 14·7737.

Qu. 32. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whoſe diameter is 289? Anſ. 278·6716.

Qu. 33. My plumber has ſet me up a ciſtern, and, his ſhop-book being burnt, he has no means of bringing in the charge, and I do not chooſe to take it down to have it weighed; but by meaſure he finds it contains $64\frac{3}{10}$ ſquare feet, and that it is preciſely $\frac{1}{8}$ of an inch in thickneſs. Lead was then wrought at 21l per fother of $19\frac{1}{2}$ cwt. It is required from theſe items to make out the bill, allowing $6\frac{5}{9}$ oz for the weight of a cubic inch of lead. Anſ. £4 11 2

Qu. 34. What will the diameter of a globe be, when the ſolidity and ſuperficial content are expreſſed by the ſame number? Anſ. 6.

Qu. 35. A ſack, that would hold 3 buſhels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another ſack contain which, being of the ſame length, has twice its breadth or circumference? Anſ. 12 buſhels.

Qu. 36. A carpenter is to put an oaken curb to a round well, at 8d per foot ſquare: the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $13\frac{1}{2}$ feet: what will be the expence? Anſ. 5s $2\frac{1}{4}$d.

Qu. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what muſt the breadth of the walk be to take up juſt half the ground? Anſ. 25·968 feet.

Qu. 38. A may-pole whoſe top, being broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the foot of the pole; what was the height of the whole may-pole, ſuppoſing the length of the broken piece to be 39 feet? Anſ. 75 feet.

Qu. 39. Seven men bought a grinding ſtone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter muſt each grind down for his ſhare?

Anſ. the 1ſt 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778.

Qu. 40. A maltſter has a kiln that is 16 feet 6 inches ſquare: but he wants to pull it down, and build a new one that may dry three times as much at once as the old one; what muſt be the length of its ſide?

Anſ. 28 f 7 inches.

Qu. 41. How many 3 inch cubes may be cut out of a 12 inch cube? Anſ. 64.

Qu. 42. How long muſt the tether of a horſe be, that will allow him to graze, quite around, juſt an acre of ground? Anſ. $39\frac{1}{4}$ yards.

Qu. 43. What will the painting of a conical ſpire come to at 8d per yard; ſuppoſing the height to be 118 feet, and the circumference of the baſe 64 feet?

Anſ. £ 14 0 $8\frac{3}{4}$.

Qu. 44. The diameter of a ſtandard corn buſhel is $18\frac{1}{2}$ inches, and its depth 8 inches; then what muſt the diameter of that buſhel be whoſe depth is $7\frac{1}{2}$ inches?

Anſ. 19·1067.

Qu. 45. Sup-

Qu. 45. Suppoſe the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it coſt at 3$\frac{1}{2}$d per ſquare inch? Anſ. £ 237 19 10$\frac{1}{2}$.

Qu. 46. What will a fruſtum of a marble cone come to at 12s per ſolid foot; the diameter of the greater end being 4 feet, that of the leſs end 1$\frac{1}{2}$, and the length of the ſlant ſide 8 feet? Anſ. £ 30 1 10$\frac{1}{4}$.

Qu. 47. To divide a cone into three equal parts by ſections parallel to the baſe, and to find the altitudes of the three parts, the height of the whole cone being 20 inches.

Anſ. the upper part 13·867
the middle part 3·604
the lower part 2·528

Qu. 48. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raiſe 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth muſt the ditch be dug, ſuppoſing its breadth to be every where 8 feet?

Anſ. 7$\frac{23}{86}$ feet.

Qu. 49. How high above the earth muſt a perſon be raiſed, that he may ſee $\frac{1}{3}$ of its ſurface?

Anſ. to the height of the earth's diameter.

Qu. 50. A cubic foot of braſs is to be drawn into wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loſs in the metal?

Anſ. 97784·797 yards, or 55 miles 984·797 yards.

Qu. 51. Of what diameter muſt the bore of a cannon be, which is caſt for a ball of 24 lb weight, ſo that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Anſ. 5·757 inches.

Qu. 52. Suppoſing the diameter of an iron 9 lb ball to be 4 inches, as it is very nearly; it is required to find the diameters of the ſeveral balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 36, and 42 lb, and the caliber of their guns, allowing $\frac{1}{50}$ of the caliber, or $\frac{1}{49}$ of the ball's diameter, for windage.

Anſwer.

Wt ball	Diameter ball	Caliber gun
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·5469	5·6601
36	6·3496	6·4792
42	6·6844	6·8208

Qu. 53. Suppoſing the windage of all mortars be allowed to be $\frac{1}{60}$ of the caliber, and the diameter of the hollow part of the ſhell to be $\frac{7}{10}$ of the caliber of the mortar: It is required to determine the diameter and weight of the ſhell, and the quantity or weight of powder requiſite to fill it, for each of the ſeveral ſorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar.

Anſwer.

Calib. mort.	Diameter ſhell	Wt ſhell empty	Wt of powder	Wt ſhell filled
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
12	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

Qu. 54. If a heavy ſphere, whoſe diameter is 4 inches, be let fall into a conical glaſs, full of water, whoſe diameter is 5, and altitude 6 inches; it is required to determine how much water will run over.

Anſ. 26·272 cubic inches, or near $\frac{35}{47}$ parts of a pint.

Qu. 55. The dimenſions of the ſphere and cone being the ſame as in the laſt queſtion, and the cone only $\frac{1}{2}$ full of water; required what part of the axis of the ſphere is immerſed in the water: Anſ. ·546 parts of an inch.

Qu. 56. The cone being ſtill the ſame, and $\frac{1}{5}$ full of water; required the diameter of a ſphere which ſhall be juſt all covered by the water. Anſ. 2·445996.

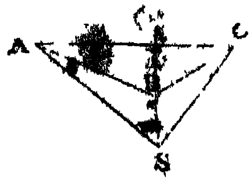
Qu. 57. If a perſon, with an air balloon, aſcend vertically from London, to ſuch height that he can juſt ſee Oxford appear in the horizon; it is required to determine his height above the earth, ſuppoſing the earth's radius to be 3965 miles, and the diſtance between London and Oxford 49·5933 miles.

Anſ. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

Qu. 58. In a garriſon there are three remarkable objects A, B, C, the diſtances of which from one to another are known to be AB 213, AC 424, and BC 262 yards; I am deſirous of knowing my poſition and diſtance at a place or ſtation S, from whence I obſerved the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Anſwer.

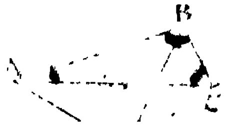
AS 605½, BS 429½, CS 524.



Qu. 59. Required the ſame as in the laſt queſtion, when the point B is on the other ſide of AC, ſuppoſing AB 9, AC 12, and BC 6 furlongs; alſo the angle ASB 33° 45′, and the angle BSC 22° 30′.

Anſwer.

AS 10·64, BS 15·64, CS 14·01.



Qu. 60. It is required to determine the magnitude of a cube of gold, of the ſtandard fineneſs, which ſhall be equal to the national debt of 240 million of pounds; ſuppoſing a guinea to weigh 5 dwts 9½ grains.

Anſ. 15·3006 feet.

Qu. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Anſ. 1158182 gallons nearly.

OF

SPECIFIC GRAVITY.

THE ſpecific gravities of bodies, are their relative weights contained under the ſame given magnitude; as a cubic foot, or a cubic inch, &c.

The ſpecific gravities of ſeveral ſorts of matter are expreſſed by the numbers annexed to their names in the following table.

A Table of the Specific Gravities of Bodies.

	03		·03
Fine gold	19640	Brick	2000
Standard gold	18888	Light earth	1984
Quick-ſilver	14000	Solid gun-powder	1745
Lead	11325	Sand	1520
Fine ſilver	11091	Pitch	1150
Standard ſilver	10535	Box-wood	1030
Copper	9000	Sea-water	1030
Gun metal	8784	Common water	1000
Caſt braſs	8000	Oak	925
Steel	7850	Gun-powder, ſhaken	922
Iron	7645	Aſh	800
Caſt iron	7425	Maple	755
Tin	7320	Elm	600
Marble	2700	Fir	550
Common ſtone	2520	Cork	240
Loom	2160	Air at a mean ſtate	$1\frac{1}{5}$

Note. The ſeveral ſorts of wood are ſuppoſed to be dry. Alſo, as a cubic foot of water weighs juſt 1000 ounces avoirdupois, the numbers in this table expreſs, not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in avoirdupois ounces; and thence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems.

PROBLEM I.

To find the Magnitude of any Body from its Weight.

As the tabular ſpecific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, reſpectively.

EXAMPLES.

Ex. 1. Required the content of an irregular block of common ſtone which weighs 1 cwt, or 112 lb.
Anſ. $1228\frac{2016}{2520}$ cub. inc.

Ex. 2. How many cubic inches of gun-powder are there in 1 lb weight. Anſ. 30 cubic inches nearly.

Ex. 3. How many cubic feet are there in a ton weight of dry oak? Anſ. $38\frac{138}{185}$ cubic feet.

PROBLEM II.

To find the Weight of a Body from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular ſpecific gravity,
To the weight of the body.

EXAMPLES.

EXAMPLES.

Ex. 1. Required the weight of a block of marble, whoſe length is 63 feet, and breadth and thickneſs each 12 feet; being the dimenſions of one of the ſtones in the walls of Balbeck.

Anſ. 683$\frac{4}{10}$ ton, which is nearly equal to the burthen of an Eaſt-India ſhip.

Ex. 2. What is the weight of 1 pint, ale meaſure, of gun-powder? Anſ. 19 oz. nearly.

Ex. 3. What is the weight of a block of dry oak, which meaſures 10 feet in length, 3 feet broad, and 2$\frac{1}{2}$ feet deep? Anſ. 4335$\frac{15}{16}$ lb.

PROBLEM III.

To find the Specific Gravity of a Body.

Case 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight loſt in water. Then

As the weight loſt in water,
Is to the whole weight,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

A piece of ſtone weighed 10 lb, but in water only 6$\frac{3}{4}$ lb, required its ſpecific gravity. Anſ. 3077.

Case 2. When the body is lighter than water, ſo that it will not quite ſink; affix to it a piece of another body heavier than water, ſo that the maſs compounded of the two may ſink together. Weigh the heavier body, and the compound maſs, ſeparately, both in water and out of it; then find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then

As this laſt remainder,
Is to the weight of the light body in air,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

Suppoſe a piece of elm weighs 15 lb in air, and that a piece of copper, which weighs 18 lb in air and 16 lb in water, is affixed to it, and that the compound weighs 8 lb in water; required the ſpecific gravity of the elm.

Anſ. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients in a given Compound.

Take the three differences of every pair of the three ſpecific gravities, namely, the ſpecific gravities of the compound and each ingredient; and multiply the difference of every two ſpecific gravities by the third. Then, as the greateſt product is to the whole weight of the compound, ſo is each of the other products to the two weights of the ingredients.

EXAMPLE.

A compoſition of 112 lb being made of tin and copper, whoſe ſpecific gravity is found to be 8784; required the quantity of each ingredient, the ſpecific gravity of tin being 7320, and of copper 9000.

$$\left.\begin{array}{l}\text{Anſwer, there is 100 lb of copper}\\ \text{and conſeq. 12 lb of tin}\end{array}\right\}\text{ in the compoſition.}$$

OF THE

WEIGHT AND DIMENSIONS

OF

BALLS AND SHELLS.

THE weight and dimenſions of balls and ſhells might be found from the problems laſt given concerning ſpecific gravity. But they may be found ſtill eaſier by means of the experimented weight of a ball of a given ſize, from the known proportion of ſimilar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9 lb, and the weights being as the cubes of the diameters, it will be as 64 (which is the cube of 4) is to 9, ſo is the cube of the diameter of any other ball, to its weight. Or take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

Ex. 1. The diameter of an iron ſhot being 6·7, required its weight. Anſ. 42·294 lb.

Ex. 2. What is the weight of an iron ball whoſe diameter is 5·54 inches? Anſ. 24 lb.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of $4\frac{1}{4}$ inches diameter weighs 17 lb; therefore as the cube of $4\frac{1}{4}$ to 17, or nearly as 9 to 2, ſo is the cube of the diameter of a leaden ball, to its weight.

Or take $\frac{2}{9}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

Ex. 1. Required the weight of a leaden ball of 6·6 inches diameter. Anſ. 63·888 lb.

Ex. 2. What is the weight of a leaden ball of 5·24 inches diameter? Anſ. 32 lb nearly.

PROBLEM III.

To find the Diameter of an Iron Ball.

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAMPLES.

Ex. 1. Required the diameter of a 42 lb iron ball. Anſ. 6·685 inches.

Ex. 2. What is the diameter of a 24 lb iron ball? Anſ. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

Multiply the weight by 9, and divide the product by 2; then the cube root of the quotient will be the diameter.

EXAMPLES.

Ex. 1. Required the diameter of a 64 lb leaden ball.
Anſ. 6·605 inches.

Ex. 2. What is the diameter of an 8 lb leaden ball?
Anſ. 3·303 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the ſhell.

That is, from the cube of the external diameter take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

Ex. 1. The outſide diameter of an iron ſhell being 12·8, and the inſide diameter 9·1 inches; required its weight.
Anſ. 188·941 lb.

Ex. 2. What is the weight of an iron ſhell, whoſe external and internal diameters are 9·8 and 7 inches?
Anſ. $84\frac{1}{4}$ lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs of powder.

EXAMPLES.

Ex. 1. How much powder will fill the ſhell whoſe internal diameter is 9·1 inches? Anſ. $13\frac{2}{11}$ lb nearly.

Ex. 2. How much powder will fill the ſhell whoſe internal diameter is 7 inches? Anſ. 6 lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

Find the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

Ex. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches. Anſ. 60 lb.

Ex. 2. How much powder will fill a cubical box whoſe ſide is 12 inches? Anſ. $57\frac{3}{5}$ lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

Multiply the ſquare of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAMPLES.

Ex. 1. How much powder will the cylinder hold whoſe diameter is 10 inches, and length 20 inches? Anſ. $52\frac{1}{3}$ lb nearly.

Ex. 2. How much powder can be contained in the cylinder, whoſe diameter is 4 inches, and length 12 inches? Anſ. $5\frac{5}{191}$ lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

Multiply the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

Ex. 1. What is the diameter of a ſhell that will hold $13\frac{1}{8}$ lb of powder? Anſ. 9·1 inches.

Ex. 2. What is the diameter of a ſhell to contain 6 lb of powder? Anſ. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box to contain a given Weight of Powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the ſide of the box in inches.

EXAMPLES.

Ex. 1. Required the ſide of a cubical box to hold 50 lb of gun-powder. Anſ. 11·44 inches.

Ex. 2. Required the ſide of a cubical box to hold 400 lb of gun-powder. Anſ. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gun-powder.

Multiply the weight in pounds by 38·2, and divide the product by the ſquare of the diameter in inches, for the length.

EXAMPLES.

Ex. 1. What length of a 36 pounder gun, of 6$\frac{2}{3}$ inches diameter, will be filled with 12 lb of powder?

Anſ. 10·314 inches.

Ex. 2. What length of a cylinder of 8 inches diameter may be filled with 20 lb of powder? Anſ. 11$\frac{15}{16}$ inc.

OF THE PILING OF BALLS AND SHELLS.

IRON balls and ſhells are commonly piled, by horizontal courſes, either in a pyramidical or wedge-like form; the baſe being either an equilateral triangle, a ſquare, or a rectangle. In the triangle and ſquare, the pile will finiſh in a ſingle ball; but in the rectangle, it will finiſh in a ſingle row of balls, like an edge.

In triangular and ſquare piles, the number of horizontal rows, or courſes, is always equal to the number of balls in one ſide of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Alſo in the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number in one ſide of the bottom row, that number increaſed by 1, and the ſame number increaſed by 2; and $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n\cdot n+1\cdot n+2}{6}$ is the number or ſum; where n is the number in the bottom row.

Ex. 1. Required the number of balls in a triangular pile, each ſide of the baſe containing 30 balls. Anſ. 4960.

Ex. 2. How many balls are in the triangular pile, each ſide of the baſe containing 20? Anſ. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

Multiply continually together the number in one ſide of the bottom courſe, that number increaſed by 1, and double the ſame number increaſed by 1; then $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n\cdot n+1\cdot 2n+1}{6}$ is the number.

EXAMPLES.

Ex. 1. How many balls are in a ſquare pile of 30 rows? Anſ. 9455.

Ex. 2. How many balls are in a ſquare pile of 20 rows? Anſ. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

From 3 times the number in the length of the baſe row, ſubtract one leſs than the breadth of the ſame, multiply the remainder by the ſaid breadth, and the product by one more than the ſame; and divide by 6 for the anſwer.

That is, $\frac{h\cdot b+1\cdot 3l-b+1}{6}$ is the number; where l is the length, and b the breadth of the loweſt courſe.

Note. In all the piles, the breadth of the bottom is equal to the number of courſes. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

Ex. 1. Required the number of balls in a rectangular pile, the length and breadth of the baſe row being 46 and 15.
Anſ. 4960.

Ex. 2. How many ſhot are in a rectangular complete pile, the length of the bottom courſe being 59, and its breadth 20? Anſ. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

From the number in the whole pile, conſidered as complete, ſubtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the fruſtum, or incomplete pile.

EXAMPLES.

Ex. 1. To find the number of ſhot in the incomplete triangular pile, one ſide of the bottom courſe being 40, and the top courſe 20. Anſ. 10150.

Ex. 2. How many ſhot are in the incomplete triangular pile, the ſide of the baſe being 24, and of the top 8?
Anſ. 2516.

Ex. 3. How many balls are in the incomplete ſquare pile, the ſide of the baſe being 24, and of the top [illegible]?
Anſ. 4760.

4. How many ſhot are in the incomplete rectangular pile, of 12 courſes, the length and breadth of the baſe being 40 and 20? Anſ. 6146.

DISTANCES

BY THE

VELOCITY OF SOUND.

BY various experiments it has been found that ſound flies, through the air, uniformly at the rate of about 1142 feet in 1 ſecond of time, or a mile in $4\frac{2}{3}$ ſeconds. And therefore, by proportion, any diſtance may be found correſponding to any given time; namely, multiply the given time, in ſeconds, by 1142, for the correſponding diſtance in feet; or take $\frac{3}{14}$ of the given time for the diſtance in miles.

Note. The time for the paſſage of ſound in the interval between ſeeing the flaſh of a gun, or lightning, and hearing the report, may be obſerved by a watch, or a ſmall pendulum. Or it may be obſerved by the beats of the pulſe in the wriſt, counting, on an average, about 70 to a minute for perſons in moderate health, or $5\frac{1}{2}$ pulſations to a mile; and more or leſs according to circumſtances.

EXAMPLES.

Ex. 1. After obſerving a flaſh of lightning, it was 12 ſeconds before I heard the thunder; required the diſtance of the cloud from whence it came. Anſ. $2\frac{4}{7}$ miles.

Ex. 2. How long, after firing the tower guns, may the report be heard at Shooters-Hill, ſuppoſing the diſtance to be 8 miles in a ſtraight line? Anſ. $37\frac{1}{3}$ ſeconds.

Ex. 3. After obſerving the firing of a large cannon at a diſtance, it was 7 ſeconds before I heard the report; what was its diſtance? Anſ. $1\frac{1}{2}$ mile.

Ex. 4. Perceiving a man at a diſtance hewing down a tree with an axe, I remarked that 6 of my pulſations paſſed between ſeeing him ſtrike and hearing the report of the blow; what was the diſtance between us, allowing 70 pulſes to a minute? Anſ. 1 mile and 198 yards.

Ex. 5. How far off was the cloud from which thunder iſſued, whoſe report was 5 pulſations after the flaſh of lightning; counting 75 to a minute? Anſ. 1523 yards.

Ex. 6. If I ſee the flaſh of a cannon fired by a ſhip in diſtreſs at ſea, and hear the report 33 ſeconds after, how far is ſhe off? Anſ. $7\frac{1}{14}$ miles.

PRACTICAL EXERCISE

IN

MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

QUESTION I. REQUIRED the weight of a caſt iron ball of 3 inches diameter, ſuppoſing the weight of a cubic inch of the metal to be 0·258 lb avoirdupois.

Anſ. 3·64739 lb.

QU. 2. To determine the weight of a hollow ſpherical iron ſhell 5 inches in diameter, the thickneſs of the metal being one inch. Anſ. 13·2387 lb.

QU. 3. Being one day ordered to obſerve how far a battery of cannon was from me, I counted by my watch 17 ſeconds between the time of ſeeing the flaſh and hearing the report; what was the diſtance? Anſ. $3\frac{1}{2}$ miles.

QU. 4. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles; required the proportion of their ſurfaces, and alſo of their ſolidities; ſuppoſing them both to be globular, as they are very nearly.

Anſ. the ſurfaces are as $13\frac{1}{2}$ to 1 nearly.
and the ſolidities as $49\frac{1}{2}$ to 1 nearly.

Qu. 5. It is propoſed to determine the proportional quantities of matter in the earth and moon; the denſity of the former being to that of the latter as 10 to 7, and their diameters as ſpecified in the preceding problem.

Anſ. as 71 to 1 nearly.

Qu. 6. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of braſs of the ſame dimenſions?

Anſ. 496 lb 14 oz.

Qu. 7. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three ſtones laid end to end, now in ſight, that meaſure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, ſo as to prepare it for moving?

Anſ. 693$\frac{9}{20}$ tons, the burthen of an Eaſt-India ſhip.

Qu. 8. The battering-ram of Veſpaſian weighed, ſuppoſe 100000 pounds; and was moved, let us admit, with ſuch a velocity, by ſtrength of hands, as to paſs through 20 feet in one ſecond of time; and this was found ſufficient to demoliſh the walls of Jeruſalem. The queſtion is with what velocity a 32 lb ball muſt move, to do the ſame execution. Anſ. 62500 feet.

Qu. 9. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the leſs moves with 1000 times the velocity of the greater: in what proportion then are the momenta or forces with which they are moved?

Anſ. the leſs moves with a force 40 times greater.

Qu. 10. A body, weighing 20 lb, is impelled by ſuch a force as to ſend it through 100 feet in a ſecond; with what velocity then would a body of 8 lb weight move, if it were impelled by the ſame force?

Anſ. 250 feet per ſecond.

Qu. 11. There are two bodies, the one of which weighs 100 lb, the other 60; but the leſs body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which theſe bodies move, is required.

Anſ. the velocity of the greater to that of the leſs, as 3 to 40.

Qu. 12. There are two bodies, the greater contains 8 times the quantity of matter in the leſs, and is moved with a force 48 times greater; the ratio of the velocities of theſe two bodies is required.

Anſ. the greater to the leſs, as 6 to 1.

Qu. 13. There are two bodies, one of which moves 40 times ſwifter than the other; but the ſwifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the ſpaces deſcribed by theſe two bodies is required.

Anſ. the ſwifter to the ſlower, as 1 to 3.

Qu. 14. Suppoſing one body to move 30 times ſwifter than another, as alſo the ſwifter to move 12 minutes, the other only 1: what difference will there be between the ſpaces deſcribed by them, ſuppoſing the laſt has moved 60 inches? Anſ. 1795 feet.

Qu. 15. There are two bodies, the one of which has paſſed over 50 miles, the other only 5; and the firſt had moved with 5 times the celerity of the ſecond: what is the ratio of the times they have been in deſcribing thoſe ſpaces? Anſ. as 2 to 1.

QU. 16. If a lever, 40 effective inches long, will, by a certain power thrown ſucceſſively upon it, in 13 hours, raiſe a weight 104 feet; in what time will two other levers, each 18 effective inches long, raiſe an equal weight 73 feet? Anſ. 10 hours $8\frac{1}{3}$ minutes.

QU. 17. What weight will a man be able to raiſe, who preſſes with the force of a hundred and a half, on the end of an equipoiſed handſpike 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine? Anſ. 2072 lb.

QU. 18. A weight of $1\frac{1}{2}$ lb laid on the ſhoulder of a man is no greater burthen to him than its abſolute weight, or 24 ounces: what difference will he feel, between the ſaid weight applied near his elbow, at 12 inches from the ſhoulder, and in the palm of his hand, 28 inches from the ſame; and how much more muſt his muſcles then draw to ſupport it at right angles, that is, having his arm ſtretched right out? Anſ. 24 lb avoirdupois.

QU. 19. What weight hung on at 70 inches from the centre of motion of a ſteel-yard, will balance a ſmall gun of $9\frac{1}{2}$ cwt, freely ſuſpended at 2 inches diſtance from the ſaid centre on the contrary ſide? Anſ. $30\frac{2}{5}$ lb.

QU. 20. It is propoſed to divide the beam of a ſteel-yard, or to find the points of diviſion where the weights of 1, 2, 3, 4, &c lb on the one ſide, will juſt balance a conſtant weight of 95 lb at the diſtance of 2 inches on the other ſide of the fulcrum, the weight of the beam being 10 lb, and its whole length 36 inches.

Anſ. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

QU. 21. Two men carrying a burthen of 200 lb weight between them, hung on a pole, the ends of which reſt on their ſhoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?

Anſ. 125 lb and 75 lb.

Qu. 22. If, in a pair of ſcales, a body weigh 90 lb in one ſcale, and only 40 lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam on each ſide of the point of ſuſpenſion.

Anſ. the weight 60 lb, and the propor. 3 to 2.

Qu. 23. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For inſtance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itſelf when the prop is 13 feet from the leſs end; but removing the prop a foot nearer to the ſaid end, it takes a man's weight of 210 lb, ſtanding on the leſs end, to hold it in equilibrium. Required the weight of the tree. Anſ. 2520 lb.

Qu. 24. If AB be a cane or walking-ſtick, 40 inches long, ſuſpended by a ſtring SD faſtened to the middle point D: now a body being hung on at E, 6 inches diſtant from D, is balanced by a weight of 2 lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2 lb weight on the other ſide is moved to G, within 8 inches of D, before the cane will reſt in equilibrio. Required the weight of the body. Anſ. 24 lb.

Qu. 25. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio upon the planes, in all poſitions of them; and what will be the altitude BD of the angle B above the horizontal plane AC when this is 50 inches long?

Anſ. BD = 24; and P to Q as AB to BC, or as 3 to 4.

Qu. 26. A lever of 6 feet long is fixed at right angles in a ſcrew, whoſe threads are one inch aſunder, ſo that the lever turns juſt once round in raiſing or depreſſing the ſcrew one inch. If then this lever be urged by a weight or force of 50 lb, with what force will the ſcrew preſs?

Anſ. $22619\frac{1}{2}$ lb.

Qu. 27. If a man can draw a weight of 150 lb up the ſide of a perpendicular wall, of 20 feet high; what weight will he be able to raiſe along a ſmooth plank of 30 feet long, laid aſlope from the top of the wall?

Anſ. 225 lb.

Qu. 28. If a force of 150 lb be applied on the head of a rectangular wedge, its thickneſs being 2 inches, and the length of its ſide 12 inches; what weight will it raiſe or balance perpendicular to its ſide?

Anſ. 900 lb.

Qu. 29. If a round pillar, of 30 feet diameter, be raiſed on a plane, inclined to the horizon in an angle of 75°, or the ſhaft inclining 15 degrees out of the perpendicular; what length will it bear before it overſet?

Anſ. $30(2 + \sqrt{3})$ or 111·9615 feet.

Qu. 30. If the greateſt angle at which a bank of natural earth will ſtand, be 45°; it is propoſed to determine what thickneſs an upright wall of ſtone muſt be made throughout, juſt to ſupport a bank of 12 feet high; the ſpecific gravity of the ſtone being to that of earth, as 5 to 4. Anſ. $12\sqrt{\frac{8}{15}}$, or 8·76356 feet.

Qu. 31. If the ſtone wall be made like a wedge, or having its upright ſection a triangle, tapering to a point at top, but its ſide next the bank of earth perpendicular to the horizon; what is its thickneſs at the bottom ſo as to ſupport the ſame bank?

Anſ. $12\sqrt{\frac{4}{5}}$, or 10·733126 feet.

Qu. 32. But if the earth will only ſtand at an angle of 30 degrees to the horizontal line; it is required to determine the thickneſs of wall in both the preceding caſes.

Anſ. the breadths are the ſame as before, becauſe the area of the triangular bank of earth is increaſed in the ſame proportion as its horizontal puſh is decreaſed.

Qu. 33. To find the thickneſs of an upright rectangular wall, neceſſary to ſupport a body of water; the water being 10 feet deep, and the wall 12 feet high; alſo the ſpecific gravity of the water, as 11 to 7. Anſ. 4·204374 feet.

Qu. 34. To determine the thickneſs of the wall at the bottom, when the ſection of it is triangular, and the altitudes as before. Anſ. 5·1492866 feet.

Qu. 35. Suppoſing the diſtance of the earth from the ſun to be 95 millions of miles, I would know at what diſtance from him another body muſt be placed, ſo as to receive light and heat quadruple to that of the earth.

Anſ. at half the diſtance, or $47\frac{1}{2}$ millions.

Qu. 36. If the mean diſtance of the ſun from us be 106 of his diameters, how much hotter is it at the ſurface of the ſun, than under our equator?

Anſ. 11236 times hotter.

Qu. 37. The diſtance between the earth and ſun being accounted 95 millions of miles, and between Jupiter and the ſun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required.

Anſ. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

Qu. 38. A

Qu. 38. A certain body on the ſurface of the earth weighs a cwt, or 112 lb; the queſtion is whither this body muſt be carried, that it may weigh only 10 lb.

Anſ. either at 3·3466 ſemi-diameters, or $\frac{5}{56}$ of a ſemi-diameter from the center.

Qu. 39. If a body weigh 1 pound or 16 ounces upon the ſurface of the earth, what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles. Anſ. 15 oz $9\frac{5}{8}$ dr nearly.

Qu. 40. Where-abouts, in the line between the earth and moon, is their common centre of gravity: ſuppoſing the earth's diameter to be 7930 miles, and the moon's 2160; alſo the denſity of the former to that of the latter, as 10 to 7, and their mean diſtance 30 of the earth's diameters?

Anſ. at $\frac{105}{251}$ parts of a diam, from the earth's center, or $\frac{61}{502}$ parts of a diameter, or 963 miles below the ſurface.

Qu. 41. Where-abouts, between the earth and moon, are their attractions equal to each other? Or, where muſt another body be placed, ſo as to remain in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimenſions being as in the laſt queſtion.

Anſ. From the earth's center $26\frac{9}{11}$ } of the earth's diameters.
From the moon's centre $3\frac{2}{11}$ }

Qu. 42. Suppoſe a ſtone, dropt into an abyſs, ſhould be ſtopped at the end of the 11th ſecond after its delivery, what ſpace would it have gone through?

Anſ. $1946\frac{1}{12}$ feet.

Qu. 43. What is the difference between the depths of two wells, into each of which ſhould a ſtone be dropped at the ſame inſtant, one will ſtrike the bottom at 6 ſeconds, the other at 10? Anſ. $1029\frac{1}{3}$ feet.

Qu. 44. If a ſtone be $19\frac{1}{2}$ ſeconds in deſcending from the top of a precipice to the bottom, what is its height? Anſ. $6115\frac{11}{16}$ feet.

Qu. 45. In what time will a muſket ball, dropped from the top of Saliſbury ſteeple, ſaid to be 400 feet high, reach the bottom? Anſ. 5 ſec. nearly.

Qu. 46. If a heavy body be obſerved to fall through 100 feet in the laſt ſecond of time, from what height did it fall, and how long was it in motion?

Anſ. time $3\frac{235}{386}$ ſec. and height $209\frac{4273}{9264}$ feet.

Qu. 47. A ſtone being let fall into a well, it was obſerved that, after being dropped, it was 10 ſeconds before the ſound of the fall at the bottom reached the ear. What is the depth of the well? Anſ. 1270 feet nearly.

Qu. 48. It is propoſed to determine the length of a pendulum vibrating ſeconds in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the firſt ſecond of time. Anſ. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

Qu. 49. What is the length of a pendulum vibrating in 2 ſeconds; alſo in half a ſecond, and in a quarter ſecond?

Anſ. the 2 ſecond pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ ſecond pendulum $9\frac{25}{32}$
the $\frac{1}{4}$ ſecond pendulum $2\frac{57}{128}$ inches.

Qu. 50. What difference will there be in the number of vibrations made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time?

Anſ. $2692\frac{1}{2}$.

Qu. 51. Obſerved that while a ſtone was deſcending to meaſure the depth of a well, a ſtring and plummet, that from the point of ſuſpenſion, or the place where it was held, to the center of oſcillation, meaſured juſt 18 inches, had made 8 vibrations. What was the depth of the well? Anſ. 412·61 feet.

Qu. 52. If a ball vibrate in the arch of a circle, 10 degrees on each ſide of the perpendicular; or a ball roll down the loweſt 10 degrees of the arch; required the velocity at the loweſt point: the radius of the circle, or length of the pendulum, being 20 inches.

Anſ. 4·4213 feet per ſec.

Qu. 53. If a ball deſcend down a ſmooth inclined plane, whoſe length is 100 feet, and altitude 10 feet; how long will it be in deſcending, and what will be the laſt velocity?

Anſ. the veloc. 25·364 feet per ſec. and time 7·8852 ſec.

Qu. 54. If a cannon ball of 1 lb weight be fired againſt a pendulous block of wood, and ſtriking the center of oſcillation, cauſe it to vibrate an arc whoſe chord is 30 inches; the radius of that arc, or diſtance from the axis to the loweſt point of the pendulum, being 118 inches, and the pendulum vibrating in ſmall arcs 40 oſcillations per minute. Required the velocity of the ball, and the velocity of the center of oſcillation of the pendulum at the loweſt point of the arc; the whole weight of the pendulum being 500 lb.

Anſ. veloc. ball 1956·6054 feet per ſec.
and veloc. cent. oſcil. 3·9054 feet per ſec.

Qu. 55. How

Qu. 55. How deep will a cube of oak ſink in common water; each ſide of the cube being 1 foot?

Anſ. $11\frac{1}{10}$ inches.

Qu. 56. How deep will a globe of oak ſink in water; the diameter being 1 foot? Anſ. 9·9867 inches.

Qu. 57. If a cube of wood, floating in common water, have 3 inches of its height dry above the water, and $4\frac{8}{10}$ inches dry when in ſea water; it is propoſed to determine the magnitude of the cube, and what ſort of wood it is made of.

Anſ. the wood is oak, and each ſide 40 inches.

Qu. 58. An irregular piece of lead ore weighs in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative denſities, or ſpecific gravities.

Anſ. as 145 to 132.

Qu. 59. An irregular fragment of glaſs in the ſcale weighs 171 grains, and another of magnet 102 grains; but in water the firſt fetches up no more than 120 grains, and the other 79: what then will their ſpecific gravities turn out to be?

Anſ. glaſs to magnet as 3933 to 5202, or nearly as 10 to 13.

Qu. 60. Hiero, king of Sicily, ordered his Jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that ſubſtituting part ſilver was only a proper perquiſite; which taking air, Archimedes was appointed to examine it; who, on putting it into a veſſel of water, found it raiſed the fluid 8·2245 cubic inches: and having diſcovered that the inch of gold more critically weighed 10·36 ounces, and that of ſilver but 5·85 ounces, he found by calculation what part of his majeſty's gold had been changed. And you are deſired to repeat the proceſs.

Anſ. 28·8 ounces.

Qu. 61. Suppoſing the cubic inch of common glaſs weigh 1·4921 ounces avoirdupois, the ſame of ſea water ·59542, and of brandy ·5368; then a ſeaman having a gallon of this liquor in a glaſs bottle, which weighs 3·84 lb out of water, and to conceal it from the officers of the cuſtoms, throws it overboard. It is propoſed to determine, if it will ſink, how much force will juſt buoy it up.

Anſ. 14·1496 ounces.

Qu. 62. Another perſon has half an anker of brandy, of the ſame ſpecific gravity as in the laſt queſtion; the wood of the caſk ſuppoſe meaſures $\frac{1}{8}$ of a cubic foot; it is propoſed to aſſign what quantity of lead is juſt requiſite to keep the caſk and liquor under water.

Anſ. 89·743 ounces.

Qu. 63. Suppoſe by meaſurement it be found that a man of war, with its ordnance, rigging, and appointments, ſinks ſo deep as to diſplace 50000 cubic feet of water; what is the whole weight of the veſſel?

Anſ. $1395\frac{1}{10}$ tons.

Qu. 64. It is required to determine what would be the height of the atmoſphere, if it were every where of the ſame denſity as at the ſurface of the earth, when the quickſilver in the barometer ſtands at 30 inches; and alſo what would be the height of a water barometer at the ſame time.

Anſ. height of the air 175000 feet or 5·5240 miles,
height of water 35 feet.

Qu. 65. With what velocity would each of thoſe three fluids, viz, quickſilver, water, and air, iſſue through a ſmall orifice in the bottom of veſſels, of the reſpective heights of 30 inches, 35 feet, and 5·5240 miles; eſtimating the preſſure by the half altitudes, and the air ruſhing into a vacuum? feet.

Anſ. the veloc. of quickſilver 8·967

the veloc. of water 33·55

the veloc. of air 968·6

But eſtimating by the whole alt. the veloc. of air is 1369·8

And the mean between theſe two is 1169·2

which is nearly the velocity of ſound, and alſo nearly equal to the velocity of a ball through the air when it ſuffers a reſiſtance equal to the preſſure of the atmoſphere.

Qu. 66. A very large veſſel of 10 feet high (no matter what ſhape) being kept conſtantly full of water, by a large ſupplying cock at the top; if 9 ſmall circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular ſide at every foot of the depth; it is required to determine the ſeveral diſtances to which they will ſpout on the horizontal plane of the baſe, and the quantity of water diſcharged by all of them in 10 minutes.

Anſ. the diſtances are

$\sqrt{18}$ or 4·24264

$\sqrt{32}$ - 5·65685

$\sqrt{42}$ - 6·48074

$\sqrt{48}$ - 6·92820

$\sqrt{50}$ - 7·07106

$\sqrt{48}$ - 6·92820

$\sqrt{42}$ - 6·48074

$\sqrt{32}$ - 5·65685

$\sqrt{18}$ - 4·24264

and the quantity diſcharged in 10 min. 87·5997 gallons.

Qu. 67. If the inner axis of a hollow globe of copper, exhauſted of air, be 100 feet; what thickneſs muſt it be of, that it may juſt float in air?

Anſ. ·02688 of an inch thick.

Qu. 68. If a ſpherical balloon of copper of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air of $\frac{1}{10}$ of the gravity of common air; what weight will juſt balance it, and prevent it from riſing up into the atmoſphere. Anſ. 20453 lb.

Qu. 69. If a glaſs tube, 36 inches long, cloſe at top, be ſunk perpendicularly into water, till its lower or open end be 30 inches below the ſurface of the water; how high will the water riſe within the tube, the quickſilver in the common barometer at the ſame time ſtanding at $29\frac{1}{2}$ inches? Anſ. 2·26545 inches.

Qu. 70. If a diving bell, of the form of a parabolic conoid, be let down into the ſea to the ſeveral depths of 5, 10, 15, and 20 fathoms; it is required to aſſign the reſpective heights to which the water will riſe within it: its axis and the diameter of its baſe being each 8 feet, and the quickſilver in the barometer ſtanding at 30·9 inches.

Anſ. at 5 fath. deep the water riſes 2·03546 feet,
at 10 - - - 3·06393
at 15 - - - 3·70267
at 20 - - - 4·14658.

PRACTICAL QUESTIONS

To exerciſe the

DOCTRINE OF FLUXIONS.

QUEST. 1. A LARGE veſſel of 10 feet high, and of any ſhape, being kept conſtantly full of water, by means of a ſupplying cock at the top; it is propoſed to aſſign the place where a ſmall hole muſt be made in the ſide of it, ſo that the water may ſpout through it to the greateſt diſtance on the plane of the baſe.

Anſ. the hole in the middle ſpouts fartheſt.

QU. 2. If the ſame veſſel ſtand on high, with its bottom a given height above a horizontal plane below; it is propoſed to determine where the ſmall hole muſt be, ſo as to ſpout fartheſt on the ſaid plane.

Anſ. in the middle between the plane and top of the veſſel.

QU. 3. But if the veſſel ſtand on an inclined plane, making an angle of 30 degrees with the horizon, it is propoſed to determine the place of the ſmall hole, ſo as to ſpout the fartheſt on the ſaid inclined plane.

Anſ. $\frac{6 + \sqrt{6}}{10} a$ below the top, the altitude of the veſſel being a.

Qu. 4. Required the ſize of a ball, which, being let fall into a conical glaſs full of water, ſhall expel the moſt water poſſible from the glaſs; its altitude being 6 inches, and diameter 5.

Anſ. the diameter of the ball $4\frac{11}{46}$ inches.

Qu. 5. It is propoſed to determine the altitude and diameter of a conical glaſs, capable of containing a pint of water, ſo that a heavy ball of 4 inches diameter, being dropt into it when full, may diſplace the moſt water poſſible.

Anſ. the altitude 4·8450
and diameter 5·2716 inches.

Qu. 6. The diſtance between two horizontal planes is a, and they are both cut at an angle of 30 degrees by an oblique plane. Query from what point in the upper plane a ball muſt be dropped, ſo that ſtriking the inclined plane, and thence rebounding from it, the ball may range the fartheſt poſſible on the lower horizontal plane; with the whole time the ball is in motion.

Anſ. height above point ſtruck on the obl. pl. $\frac{2 \mp \sqrt{3}}{3} a$,
the greateſt range - - - - — a or $2a$,
the whole time $\sqrt{\frac{\frac{2}{3} a}{16\frac{1}{12}}} + \sqrt{\frac{\frac{2 + \sqrt{3}}{3} a}{16\frac{1}{12}}} \doteq 4{\cdot}8171$ when $a = 100$.

Qu. 7. If an elaſtic ball be let fall from the height of 100 feet above the plane of the baſe of a hemiſphere, of 10 feet diameter, lying upon a horizontal plane; it is propoſed to determine on what part of the hemiſphere the ball muſt impinge, ſo that it may thence rebound to the greateſt diſtance on the plane; alſo to aſſign the extent of that diſtance, and the whole time in motion.

Anſ. ſtrikes the hemiſ. at 4·0112 above the baſe,
the greateſt diſtance 114·2846 without the hemiſph.
and the whole time 4·16705″ in motion.

And if one of the forces, as F, be the force of gravity at the ſurface of the earth, and be called 1, and its time T be = 1″; then it is known by experiment that the correſponding ſpace S is = $16\frac{1}{12}$ feet, and conſequently its velocity V = 2S = $32\frac{1}{6}$, which call $2g$, namely $g = 16\frac{1}{12}$ feet or 193 inches. Then the above four theorems, in this caſe, become as follows:

$$5.\quad s = \frac{tv}{2} = gft^2 = \frac{v^2}{4gf}.$$

$$6.\quad v = 2gft = \frac{2s}{t} = \sqrt{4gfs}.$$

$$7.\quad t = \frac{v}{2gf} = \frac{2s}{v} = \sqrt{\frac{s}{gf}}.$$

$$8.\quad f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}.$$

And from theſe are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces,*

$$9.\quad s = \frac{vv}{2gf} = v\dot{t}.$$

$$10.\quad \dot{v} = \frac{2gf\dot{s}}{v} = 2gf\dot{t}.$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}$$

$$12.\quad f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}.$$

In theſe four theorems, the force f, though variable, is ſuppoſed to be conſtant for the indefinitely ſmall time $\dot{t}$; and they are to be uſed in all caſes of variable forces, as the former ones in conſtant forces; namely, from the circumſtances of the problem under conſideration, deduce an expreſſion for the value of the force f, which ſubſtitute in one of theſe theorems, which ſhall be proper to the caſe in hand, and the equation thence reſulting will determine the correſponding values of the other quantities required in the problem.

Note. That the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it; and, therefore, m will be found by ſubſtituting $\frac{m}{q}$ for f in the theorems above.

Alſo the momentum, or quantity of motion, is qv, the product of the velocity and matter.

It is alſo to be obſerved, that the theorems equally hold good for the deſtruction of motion and velocity, by means of retarding forces, as for the generation of the ſame by means of accelerating forces.

And to the following problems, which are all reſolved by the application of theſe theorems, it has been thought proper to ſubjoin their ſolutions, for the better information and convenience of the ſtudent.

The following table of the moſt common and uſeful forms of fluxions and fluents, is inſerted, as it will be found very uſeful for aſſigning the fluents found in many of the following problems.

Note. The logarithms mentioned in the fluents, are the hyperbolic ones, and the radius of the circle is 1.

Forms	Fluxions	Fluents
I	$x^{n-1}\dot{x}$	$\frac{1}{n}x^{n}$ This form fails when $n = 0$
II	$\overline{a \pm x^{n}}^{\,m-1}\; x^{n-1}\dot{x}$	$\frac{\pm 1}{mn} \times \overline{a \pm x^{n}}^{\,m}$ Fails when m or $n = 0$
III	$\frac{x^{mn-1}\dot{x}}{\overline{a \pm x^{n}}^{\,m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{\overline{a \pm x^{n}}^{\,m}}$ Fails when a or m or $n = 0$
IV	$x^{-1}\dot{x}$ or $\frac{\dot{x}}{x}$	log. x
V	$\frac{x^{n-1}\dot{x}}{a \pm x^{n}}$	$\pm\frac{1}{n}$ log. $a \pm x^{n}$ Fails when $n = 0$
VI	$\frac{x^{-1}\dot{x}}{a \pm x^{n}}$	$\frac{1}{an}$ log. $\frac{x^{n}}{a \pm x^{n}}$
VII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^{n}}$	$\frac{1}{n\sqrt{a}}$ log. $\frac{\sqrt{a}+\sqrt{x^{n}}}{\sqrt{a}-\sqrt{x^{n}}}$
VIII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a + x^{n}}$	$\frac{1}{n\sqrt{a}} \times 2$ arc tang. $\sqrt{\frac{x^{n}}{a}}$ or $\frac{1}{n\sqrt{a}} \times$ arc coſin. $\frac{a - x^{n}}{a + x^{n}}$
IX	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{\pm a + x^{n}}}$	$\frac{2}{n}$ log. $\sqrt{x^{n}} + \sqrt{\pm a + x^{n}}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a - x^{n}}}$	$\frac{1}{n} \times 2$ arc ſin. $\sqrt{\frac{x^{n}}{a}}$ or $\frac{1}{n} \times$ arc vers. $\frac{2x^{n}}{a}$
XI.	$\frac{x^{-1}\dot{x}}{\sqrt{a \pm bx^{n}}}$	$\frac{1}{n\sqrt{a}}$ log. $\frac{\pm\sqrt{a \pm x^{n}} \mp \sqrt{a}}{\sqrt{a \pm x^{n}} + \sqrt{a}}$
XII	$\frac{x^{-1}\dot{x}}{\sqrt{-a + x^{n}}}$	$\frac{1}{n\sqrt{a}} \times 2$ arc ſec. $\sqrt{\frac{x^{n}}{a}}$ or $\frac{1}{n\sqrt{a}} \times$ arc coſin. $\frac{2a - x^{n}}{x^{n}}$

PROBLEM I.

To determine the time and velocity of a body deſcending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by mechanics, 20 : 1 :: 1, the force of gravity : $\frac{1}{20} = f$, the force on the plane.

Theref. by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20}$ $= \sqrt{4 \times 16\frac{1}{12}} = 2 \times 4\frac{1}{96} = 8\frac{1}{48}$ feet, the laſt velocity per ſecond,

And, by theor. 7, t or $\sqrt{\frac{s}{gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}}$ $= \frac{20}{4\frac{1}{96}} = 4\frac{76}{77}$ ſeconds, the time of deſcending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per ſecond, in the direction of, and up, a ſmooth inclined plane, that ariſes 1 foot in 20: it is propoſed to aſſign the length which it will aſcend up the plane before it ſtops and begins to return down again, and the time of its aſcent.

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}}$ $\frac{60000000}{193} = 310880\frac{160}{193}$ feet, or nearly 59 miles, the diſtance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193}$ $= 621''\frac{147}{193} = 10'\ 21''\frac{147}{193}$, the time of aſcent.

PROBLEM III.

If a ball be projected up a ſmooth inclined plane, which riſes 1 foot in 10, and aſcend 100 feet before it ſtop: required the time of aſcent, and the velocity of projection.

Firſt, by theor. 6, $v = \sqrt{4gfs} = \sqrt{4 \times 16\tfrac{1}{12} \times \tfrac{1}{10} \times 20} = 8\tfrac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per ſecond, the velocity.

And, by theor. 7, $t = \sqrt{\dfrac{s}{gf}} = \sqrt{\dfrac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \dfrac{10}{4\frac{1}{96}}\sqrt{10}$ $= \tfrac{192}{77}\sqrt{10} = 7{\cdot}88516$ ſeconds, the time in motion.

PROBLEM IV.

If a ball be obſerved to aſcend up a ſmooth inclined plane, 100 feet in ten ſeconds, before it ſtop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

Firſt, by theor. 6, $v = \dfrac{2s}{t} = \dfrac{200}{10} = 20$ feet per ſec. the velocity.

And, by theor. 8, $f = \dfrac{s}{gt^2} = \dfrac{100}{16\frac{1}{12} \times 100} = \dfrac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore, 193 : 12 :: 100 : 6·2176 the height of the plane, or the ſine of elevation to radius 100, which anſwers to 3° 34′, the angle of elevation of the plane.

PROBLEM V.

By a mean of ſeveral experiments, I have found that a caſt iron ball, of 2 inches diameter, impinging perpendicularly on the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per ſecond, penetrated 13 inches deep into its ſubſtance. It is propoſed then to determine the time of the penetration, and the reſiſting force of the wood as compared to the force of gravity, ſuppoſing that force to be a conſtant quantity. Firſt, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a ſecond, the time.

And, by th. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is the reſiſting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its denſity or ſpecific gravity. For, ſince f is as $\frac{v^2}{s}$ by theor. 4, the denſity and ſize of the ball remaining the ſame; if the denſity, or ſpecific gravity, n, vary, and all the reſt be conſtant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the ſize of the ball only is conſtant. But when only the diameter d varies, all the reſt being conſtant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the reſiſting ſurface, or force of reſiſtance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the reſt are conſtant. Conſequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And ſo $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$, and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$;

where f denotes the ſtrength or firmneſs of the ſubſtance penetrated, and is here ſuppoſed to be the ſame, for all balls and velocities, in the ſame ſubſtance, which it is either accurately or nearly ſo. See pa. 264 &c. of my Tracts, vol. 1.

Hence, taking the numbers in the problem, it is $f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$ the value of f for elm wood. Where the ſpecific gravity of the ball is taken $7\frac{1}{3}$, which is a little leſs than that of ſolid caſt iron, as it ought, on account of the air bubble that is caſt in all balls.

PROBLEM VI.

To find how far a 24 lb ball of caſt iron will penetrate into a block of ſound elm, when fired with a velocity of 1600 feet per ſecond.

Here, becauſe the ſubſtance is the ſame as in the laſt problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5{\cdot}55 \times 1600^2 \times 13}{2 \times 1500^2}$ $= 41\frac{1}{2}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. 1 pa. 273) that an 18 pounder ball, fired with a velocity of 1200 feet per ſecond, penetrated 34 inches into round dry oak. It is required then to aſcertain the comparative ſtrength or firmneſs of oak and elm.

The diameter of an 18 lb ball is 5·04 inches = D. Then, by the numbers given in this problem for oak, and in prob. v for elm, we have $\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5{\cdot}04 \times 1200^2 \times 13}$ $= \frac{100 \times 17}{5{\cdot}04 \times 16 \times 13} = \frac{1700}{1048}$ or $= \frac{7}{5}$ nearly.

From which it would ſeem that elm reſiſts more than oak, in the ratio of about 7 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be ſuſpected that Mr. Robins's great penetration was owing to the ſplitting of his timber in ſome degree.

PROBLEM VIII.

A 24 pounder ball being fired into a bank of firm earth, with a velocity of 1300 feet per ſecond, penetrated 15 feet: It is required then to aſcertain the comparative reſiſtances of elm and earth.

Comparing the numbers here with thoſe in prob. 5, it is $\frac{f}{\text{F}} = \frac{dv^2s}{\text{D}\text{V}^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5{\cdot}55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^2 \times 0{\cdot}37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm reſiſts about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; ſuppoſing it fired with a velocity of 1700 feet per ſecond, and that the lead does not change its figure by the ſtroke againſt the wood.

Here $\text{D} = \frac{3}{4}$, $\text{N} = 11\frac{1}{3}$, $n = 7\frac{1}{5}$. Then by the numbers and theorem in prob. 5, it is $\text{S} = \frac{\text{DNV}^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{5} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber muſt have reſiſted about twice as much; or elſe, which is very probable, that the defect in his penetration aroſe from the change of figure in the leaden ball from the blow againſt the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per ſecond, having been found to penetrate 13 inches deep into dry elm: It is required to aſcertain the time of paſſing through every ſingle inch of the 13, and the velocity loſt at each of them; ſuppoſing the reſiſtance of the wood conſtant or uniform.

The velocity v being 1500 feet, or 1500 × 12 = 9000 inches, and velocities and times being as the roots of the ſpaces, in conſtant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a ſecond, the whole time of paſſing through the 13 inches; therefore as

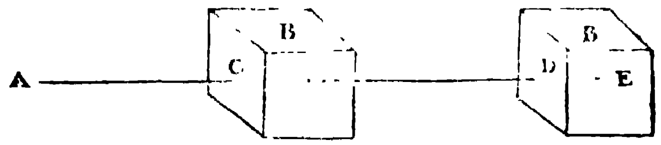
$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

	veloc. loſt		Time	In the
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}v =$	58·8 :: t :	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}t =$	·00005	1ſt
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}v =$	61·4 :: t :	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}t =$	·00006	2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}v =$	64·2 &c	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}t =$	·00006	3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}v =$	67·5	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}t =$	·00007	4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}v =$	71·4	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}t =$	·00007	5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}v =$	76·0	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}t =$	·00007	6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}v =$	81·6	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}t =$	·00008	7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}v =$	88·8	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}t =$	·00008	8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}v =$	98·3	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}t =$	·00009	9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}v =$	111·4	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}t =$	·00011	10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}v =$	132·3	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}t =$	·00013	11th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}v =$	172·3	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}t =$	·00017	12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}v =$	416·0	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}t =$	·00040	13th
Sum	1500·0	Sum $\frac{1}{692}$ or	·00144	inch

Hence, as the motion loſt at the beginning is very ſmall; and conſequently the motion communicated to any body, as an inch plank, in paſſing through it, is very ſmall alſo; we can conceive ſuch a plank ſhot through, without overſetting it; and may eaſily compute the height and breadth of ſuch a plank as will juſt ſtand without being overſet by the ball in paſſing through it.

PROBLEM XI.

To determine the circumſtances of ſpace, penetration, velocity, and time, ariſing from a ball moving with a given velocity, and ſtriking a moveable block of wood, or other ſubſtance.



Let the ball move in the direction AE paſſing through the centre of gravity of the block B, impinging on the point C; and when the block has moved through the ſpace CD, in conſequence of the blow, let the ball have penetrated to the depth DF.

Let B = the maſs or matter in the block,
b = the ſame in the ball,
s = CD the ſpace moved by the block,
x = DE the penetration of the ball, and theref.
$s + x$ = CE the ſpace deſcribed by the ball,
a = the firſt velocity of the ball,
v = the velocity of the ball at E,
u = veloc. of the block at the ſame inſtant,
t = the time of penetration, or of the motion,
r = the reſiſting force of the wood.

Then ſhall $\frac{r}{\text{B}}$ be the accelerating force of the block,

and $\frac{r}{b}$ the retarding force of the ball.

Now

Now becauſe the momentum $B\dot{u}$, communicated to the block in the time $\dot{t}$, is that which is loſt by the ball, namely $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = a$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum loſt by the ball. And when the ball has penetrated to the utmoſt depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B + b)u$; that is, the momentum before the ſtroke, is equal to the momentum after it. And the velocity communicated will be the ſame, whatever be the reſiſting force of the block, the weight being the ſame.

Again, by theor. 6, it is $u^2 = \frac{4grs}{B}$, and $-v^2 = \frac{4gr}{b} \times (s + x)$, or rather, by correction, $a^2 - v^2 = \frac{4gr}{b}(s + x)$. Hence the penetration or $x = \frac{b(a^2 - v^2) - 4grs}{4gr}$. And when $v = u$, by ſubſtituting u for v, and Bu^2 for $4grs$, the greateſt penetration becomes $\frac{ba^2 - (B + b)u^2}{4gr}$; and this again, by writing ab for its value $(B + b)u$, gives the greateſt penetration $x = \frac{Bba^2}{4gr(B + b)} = \frac{ba^2}{4gr} \times (1 - \frac{b}{B + b})$. Which is barely equal to $\frac{ba^2}{4gr}$ when the block is fixed, or infinitely great; and is always very nearly equal to the ſame $\frac{ba^2}{4gr}$ when B is very great in reſpect of b. Hence

$$s + x = \frac{a^2 - u^2}{4gr} = \frac{a^2 - \frac{a^2b^2}{(B + b)^2}}{4gr} = \frac{B^2 + 2Bb}{(B + b)^2} \times \frac{a^2b}{4gr}$$

And therefore $B + b : B + 2b :: x : s + x$,

or $B + b : b :: x : s$,

and $s = \frac{bx}{B + b} = \frac{Bb^2a^2}{4gr(B + b)^2}$.

Ex. When the ball is iron, and weighs 1 pound, it penetrates elm about 13 inches when it moves with a velocity of 1500 feet per ſecond, in which caſe,

$$\frac{r}{b} = \frac{a^2}{4gx} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{9000^2}{193 \times 13} = 32284$$

nearly.

When B = 500 lb, and $b = 1$; then $u = \frac{ab}{\text{B} + b} =$

$$\frac{1500}{501} = 3$$ feet nearly per ſecond, the velocity of the block.

Alſo $s = \frac{\text{B}u^2}{4gr} = \frac{500 \times 9}{4 \times 16\frac{1}{12} \times 32284} = \frac{1}{461\frac{1}{2}}$ part of a foot, or $\frac{2}{77}$ of an inch, which is the ſpace moved by the block when the ball has completed its penetration.

And $t = \frac{2s}{u} = \frac{2}{461\frac{1}{2} \times 3} = \frac{1}{692}$ part of a ſecond,

or $$t = \frac{2s + 2x}{v} = \frac{\frac{2}{461\frac{1}{2}} + \frac{26}{12}}{1500} = \frac{6 + 13\;231}{6.231.1500} = \frac{1'}{692}$$

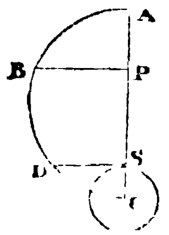
part of a ſecond, the time of penetration.

For the circumſtances relating to the motion of a block, hung by, and vibrating on, an axis, when ſtruck by a ball, ſee my Tracts, pa. 116, &c.

PROBLEM XII.

The force of attraction, above the earth, being inverſely as the ſquare of the diſtance from the centre; it is propoſed to determine the time, velocity, and other circumſtances, attending a heavy body falling from any given height; the deſcent at the earth's ſurface being $16\frac{1}{12}$ feet, or 193 inches, in the firſt ſecond of time.

Put



r = CS the radius of the earth,
a = CA the diſt. fallen from,
x = CP any variable diſtance,
v = the velocity at P,
t = time of falling there, and
$g = 16\frac{1}{12}$, half the veloc. or force at A,
f = the force at the point P.

Then we have the three following equations,

$x^2 : r^2 :: 2g : f = \frac{2gr^2}{x^2}$ the force at P, or the velocity per ſecond that would be generated by the force there,

$$\dot{t}v = -\dot{x}, \text{ and}$$

$$vv = -fx = -\frac{2gr^2\dot{x}}{x^2}$$

The fluents of the laſt equation give $v^2 = \frac{4gr^2}{x}$.

But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{\frac{4gr^2}{a} \times \frac{a-x}{x}}$, a general expreſſion for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{4gr \times \frac{a-r}{a}}$ for the greateſt velocity, or the velocity when the body ſtrikes the earth.

When a is very great in reſpect of r, the laſt velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greateſt velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per ſecond. Alſo the velocity acquired in falling from the diſtance of the ſun, or 12000 diameters of the earth, is 6·9505 miles per ſecond. And the velocity acquired in falling from the diſtance of the moon, or 30 diameters, is 6·8924 miles per ſecond.

Again, to find the time, ſince $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} +$ arc to diameter a and vers. $a - x)$; or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the ſurface at S; which is evidently infinite when a or AC is infinite, although the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's ſurface is given $= g$; becauſe r is then nearly $= a$, and AD nearly $=$ DS, the time t for the diſtance g will be nearly $\sqrt{\frac{r}{4gr^2}} \times 2\text{DS}$ $= \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

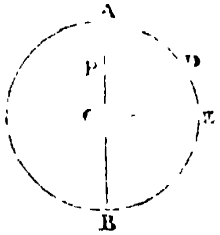
If a body at the diſtance of the moon at A fall to the earth's ſurface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$da. 19h. 46′ 46″, the time of falling from the moon to the earth.

When the attracting body is conſidered as a point C; the whole time of deſcending to C will be
$\frac{1}{2r}\sqrt{\frac{a}{g}} \times \text{ABDC} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{g}}$.

PROBLEM XIII.

The force of attraction below the earth's ſurface being directly as the diſtance from the center; it is propoſed to determine the circumſtances of velocity, time, and ſpace fallen by a heavy body from the ſurface, through a perforation made ſtraight to the centre of the earth; abſtracting from the effect of the earth's rotation, and ſuppoſing it to be a homogeneous ſphere of 3965 miles radius.

Put $r =$ AC the radius of the earth,
$x =$ CP the diſtance fallen,
$v =$ the velocity at P,
$t =$ the time there,
$g = 16\tfrac{1}{12}$, half the force at A,
$f =$ the force at P.



Then CA : CP :: $2g : f$; and the three equations are $rf = 2gx$, and $v\dot{v} = -f\dot{x}$, and $\dot{t}v = -\dot{x}$. Hence $f = \frac{2gx}{r}$, and $v\dot{v} = \frac{-gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{2g \times \frac{r^2 - x^2}{r}} = \text{PD}\sqrt{\frac{2g}{r}} = \text{PD}\sqrt{\frac{2g}{\text{CE}}}$ the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{2g \cdot \text{AC}} = 25950$ feet or 4·9148 miles per ſecond, which is the greateſt velocity, or that at the centre C.

Again, for the time, $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$, and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to coſine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of deſcent to any point P, is as the correſponding arc AD.

When P arrives at C, the above becomes $t = \sqrt{\frac{1}{2gr}}$ × quadrant AE $= \frac{AE}{AC}\sqrt{\frac{r}{2g}} = 1\cdot5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ ſeconds $= 21'\ 7''\frac{1}{4}$, for the time of falling to the centre C.

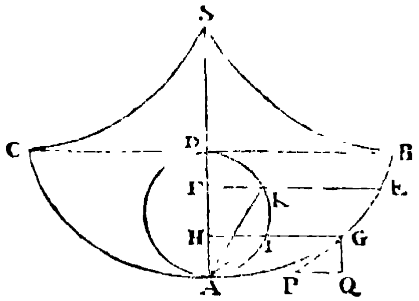
The time of falling to the centre is the ſame quantity $1\cdot5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For let n be any given diſtance from C at which the motion commences: then, by correction, $v = \sqrt{\frac{2g}{r}(n^2 - x^2)}$; and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}}$ × arc to coſine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1\cdot5708\sqrt{\frac{r}{2g}}$ for the time of deſcent to the centre C.

As an equal force, acting in contrary directions, generates or deſtroys an equal quantity of motion, in the ſame time; it follows that, after paſſing the centre, the body will juſt aſcend to the oppoſite ſurface at B, in the ſame time in which it fell to the centre from A; then from B it will return again in the ſame manner, through C to A; and ſo vibrate continually between A and B, the velocity being always equal at equal diſtances from C on both ſides; and the whole time of a double oſcillation, or of paſſing from A and arriving at A again, will be quadruple the time of paſſing over the radius AC, or =

$2 \times 3\cdot1416\sqrt{\frac{r}{2g}} = 1^{h}\ 24'\ 29''$.

PROBLEM XIV.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let S be the point of ſuſpenſion,

SA = the arc SB or SC the length of the pendulum,
CA = AB = SB or SC the ſemi-cycloid,
AD = DS the diameter of its generating circle, to which FKE, HIG are perpendiculars.

To any point G draw the tangent GP, alſo draw GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direct. GP :: GP : GQ :: AI : AH :: AD : AI; in like manner, the force of grav. : force in curve at E :: AD : AK; that is, the accelerative force in the curve, is as the correſponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, ſince AG is always = 2 AI. So that the proceſs and concluſions for the velocity and time of deſcribing any arc in this caſe, will be the ſame as in the laſt problem.

From whence it follows that the time of a ſemi-vibration in all arcs, AG, AE, &c, is the ſame conſtant quantity $1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{\mathrm{A}}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$, and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{2g}}$, where $l = \mathrm{AS} = \mathrm{AB}$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3·1416 the circumference of a circle whoſe diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, to the time of one vibration.

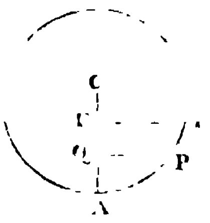
If the time of the whole vibration be 1 ſecond, this equation ariſes, $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$, and hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348\,l$. So that if one of theſe, g or l, be given by experiment, theſe equations will give the other. When g, for inſtance, is ſuppoſed to be $16\frac{1}{12}$ feet, or 193 inches, then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$ the length of a pendulum to vibrate ſeconds. Or if $l = 39\frac{1}{8}$, the length of the ſeconds pendulum for the latitude of London, then is $g = 4{\cdot}9348\,l = 193{\cdot}07$ inches $= 16\frac{107}{1200}$ feet, or nearly $16\frac{1}{12}$ feet, for the ſpace deſcended by gravity in the firſt ſecond of time in the latitude of London.

Hence the times of vibration of pendulums, are as the ſquare roots of their lengths; and the number of vibrations made in a given time, reciprocally as the ſquare roots of the lengths. And hence the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc, as the length of the ſtring is conſtantly the ſame, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is ſmaller; ſo that when it is very ſmall, the times of vibration will be nearly equal. And hence $39\frac{1}{8}$ inches is the length of a pendulum vibrating ſeconds in the very ſmall arc of a circle.

PROBLEM XV.

To find the Velocity and Time of a Heavy Body deſcending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.



Let D be the beginning of the deſcent, C the centre, and A the loweſt point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the ſame as in Q by falling through EQ, it will be $v = 2\sqrt{g \times \text{EQ}} = 2\sqrt{g(a - x)}$, when $a = \text{AE}$, $x = \text{AQ}$, and $g = 16\frac{1}{12}$.

But the fluxion of the time $\dot{t}$ is $= \frac{-\dot{\text{AP}}}{v}$, and $\dot{\text{AP}} = \frac{r\dot{x}}{\sqrt{2rx - x^2}}$, where $r =$ the radius AC. Therefore

$$\dot{t} = \frac{r}{2\sqrt{g}} \times \frac{-\dot{x}}{\sqrt{2rx - x^2} \times \sqrt{a - x}} = \frac{d}{4\sqrt{g}} \times \frac{\quad}{\sqrt{ax - x^2} \times \sqrt{d - x}},$$

where $d = 2r$ the diameter.

Or $\dot{t} = \frac{1}{4}\sqrt{\frac{d}{g}} \times \frac{-\dot{x}}{\sqrt{ax-x^2}}\left(1 + \frac{x}{2d} + \frac{1.3x^2}{2.4d^2} + \frac{1.3.5x^3}{2.4.6d^3}\right.$ &c).

But the fluent of $\frac{\dot{x}}{\sqrt{ax-x^2}}$ is $\frac{2}{a} \times$ arc to rad. $\frac{1}{2}a$ and vers. x, or it is the arc whoſe rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the ſucceeding terms, without the coefficients, be B, C, D, E, &c. Then will the fluxion of any one, as $\dot{\mathrm{Q}}$, at n diſtance from A, be $\dot{\mathrm{Q}} = x^n\dot{\mathrm{A}} = x\dot{\mathrm{P}}$, which ſuppoſe alſo = the fluxion of $b\mathrm{P} - dx^{n-1}\sqrt{ax-x^2} = b\dot{\mathrm{P}} - d\cdot\overline{n-1}\cdot\dot{x}x^{n-2}\sqrt{ax-x^2}$ $- d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax-x^2}{\sqrt{ax-x^2}} = b\dot{\mathrm{P}} - d\dot{x} \times \frac{\overline{n-\frac{1}{2}}\cdot ax^{n-1} - nx^n}{\sqrt{ax-x^2}}$ $= b\dot{\mathrm{P}} - d\cdot\overline{n-\frac{1}{2}}\cdot a\dot{\mathrm{P}} + dnx\dot{\mathrm{P}}$.

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n-1}{2n}a$; and $\mathrm{Q} = \frac{\overline{2n-1}\,.\,a\mathrm{P} - 2x^{n-1}\sqrt{ax-x^2}}{2n}$. Which being ſubſtituted, the fluential terms become $\frac{1}{4}\sqrt{\frac{d}{g}} \times \left(-\mathrm{A} - \frac{1}{2d}\cdot\frac{a\mathrm{A} - 2\sqrt{ax-x^2}}{2} - \frac{1\cdot3}{2\cdot4d^2}\cdot\frac{3a\mathrm{B} - 2x\sqrt{ax-x^2}}{4} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6d^3}\cdot\frac{5a\mathrm{C} - 2x^2\sqrt{ax-x^2}}{6}\right.$ − &c).

But when $x = a$, thoſe terms become barely

$$\frac{3\cdot1416}{4}\sqrt{\frac{d}{g}} \times \left(-1 - \frac{1^2a}{2^2d} - \frac{1^2.3^2a^2}{2^2.4^2d^2} - \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3} - \right.$$

&c); which being ſubtracted, and x taken = 0, there ariſes for the whole time of deſcending down DA, or the corrected value of $t = \frac{3\cdot1416}{4}\sqrt{\frac{d}{g}} \times \left(1 \cdot \frac{1^2a}{2^2d} \cdot \frac{1^2.3^2a^2}{2^2.4^2d^2} + \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3} + \right.$ &c).

When the arc is ſmall, as in the vibration of the pendulum of a clock, all the terms of the ſeries may be omitted after the ſecond, and then the time of a vibration t is nearly $= 1{\cdot}5708\sqrt{\frac{r}{2g}} \times (1 + \frac{a}{8r})$. And therefore the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the verſed ſine of the arc.

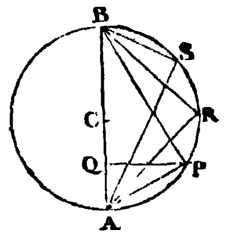
If D be the degrees of the pendulum's vibration, on each ſide of the loweſt point of the ſmall arc, the radius being r, the diameter d, and $3{\cdot}1416 = p$; then is the length of that arc $A = \frac{prD}{180}$. But the verſed ſine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r^3} + \&c = \frac{A^2}{d} - \frac{A^4}{3d^3} + \&c$. Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4} + \&c = \frac{p^2D^2}{360^2} - \frac{p^4D^4}{360^4} + \&c$, or only $= \frac{p^2D^2}{360^2}$ the firſt term, by rejecting all the reſt of the terms on account of their ſmallneſs, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D^2}{12787}$. This value then being ſubſtituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the laſt near value of the time, it becomes $t = 1{\cdot}5708\sqrt{\frac{r}{2g}} \times (1 + \frac{D^2}{51150})$ nearly. And therefore the times of vibration in different ſmall arcs, are as $51150 + D^2$, or as 51150 added to the ſquare of the number of degrees in the arc.

Hence it follows that the time loſt in each ſecond, by vibrating in a circle, inſtead of the cycloid, is $\frac{D^2}{51150}$, and conſequently the time loſt in a whole day of 24 hours, or $24 \times 60 \times 60$ ſeconds, is $\frac{5}{3}D^2$ nearly. In like manner, the ſeconds loſt per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$. Therefore, if the pendulum keep true time in one of theſe arcs, the ſeconds loſt or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum meaſure true time in an arc of 3 degrees, it will loſe $11\frac{2}{3}$ ſeconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ ſeconds a day by vibrating 5 degrees; and ſo on.

And in like manner, we might proceed for any other curve, as the ellipſe, hyperbola, parabola, &c.

PROBLEM XVI.

To determine the Time of a Body deſcending down the Chord of a Circle.



Let C be the centre, AB the vertical diameter, AP any chord down which a body is to deſcend from P to A, and PQ perpendicular to AB. Now as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular diſtances below PQ; and conſequently the time in PA : time in QA :: PA : QA :: BA : PA; but time in BA : time in QA :: $\sqrt{BA}$: $\sqrt{QA}$ :: BA : PA; hence, as three of the terms in each proportion are the ſame, the fourth terms muſt be equal, namely the time in BA = the time PA.

And in like manner the time in BP = the time in BA. So that, in general, the times of deſcending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and $r =$ the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

SCHOLIUM. By comparing this with the reſults of the two preceding problems, it will appear that the times in the cycloid, and in the arc of a circle, and in any chord of the circle, are reſpectively as the three quantities

$$1,\ 1 + \frac{a}{8r}\ \&c, \text{ and } \frac{1}{\cdot 7854}$$

or nearly as the three quantities 1, $1 + \frac{a}{8r}$, $1{\cdot}27324$; the firſt and laſt being conſtant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Alſo the time in the cycloid is the leaſt, but in the chord the greateſt; for the greateſt value of the ſeries, in prob. 15, when $a = r$, or the arc AD is a quadrant, is $1{\cdot}18014$; and in that caſe the proportion of the three times is as the numbers 1, $1{\cdot}18014$, $1{\cdot}27324$. Moreover the time in the circle approaches to that in the cycloid, as the arc decreaſes, and they are very nearly equal when that arc is very ſmall.

PRO-

PROBLEM XVII.

To find the Time and Velocity of a Chain, conſiſting of very ſmall links, deſcending from a ſmooth horizontal plane; the Chain being 100 inches long, and 1 Inch of it hanging off the Plane at the Commencement of Motion.

Put $a = 1$ inch, the length at the beginning;
$l = 100$ the whole length of the chain;
$x =$ any variable length off the plane.

Then x is the motive force to move the body, and $\frac{x}{l} = f$ the accelerative force.

Hence $v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}$.

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, therefore, by correction, $v^2 = 2g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{2g \times \frac{x^2 - a^2}{l}}$ the velocity for any length x. And when the chain juſt quits the plain, $x = l$, and then the greateſt velocity is $\sqrt{2g \times \frac{l^2 - a^2}{l}} = \sqrt{2 \times 193 \times \frac{100^2 - 1^2}{100}} = \sqrt{\frac{386 \times 9999}{100}} = 196{\cdot}45902$ inches, or 16·371585 feet, per ſecond.

Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$; the correct fluent of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x + \sqrt{x^2 - a^2}}{a}$, the time for any length x. And when $x = l = 100$, it is $t =$

$\sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9099}}{1} = 2{\cdot}69676$ ſeconds, the time when the laſt of the chain juſt quits the plane.

PROBLEM XVIII.

To find the Time and Velocity of a Chain, of very ſmall Links, quitting a Pulley, by paſſing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the Beginning.

Put $a = 2$, $l = 200$, and $x =$ BD any variable difference of the two parts AB, AC.

Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g \cdot \frac{x}{l} \cdot \frac{1}{2}\dot{x} = \frac{gx\dot{x}}{l}$.

Hence the correct fluent is $v^2 = g \times \frac{x^2 - a^2}{l}$, and

$v = \sqrt{g \times \frac{x^2 - a^2}{l}}$, the general expreſſion of the velocity. And when $x = l$, or C arrives at A, it is $v = \sqrt{g \times \frac{l^2 - a^2}{l}} = \sqrt{193 \times \frac{200^2 - 2^2}{200}} = \sqrt{386 \times \frac{100^2 - 1^2}{200}} = \sqrt{\frac{386 \times 9999}{100}} = 196{\cdot}45902$ inches, or $16{\cdot}371585$ feet, for the greateſt velocity when the chain juſt quits the pulley.

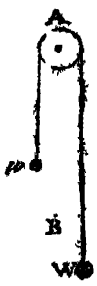
Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$. And the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x + \sqrt{x^2 - a^2}}{a}$, the general expreſſion for the time. And when $x = l$, it becom[illegible] $= \sqrt{\frac{l}{4g}} \times \log. \frac{l + \sqrt{l^2 - a^2}}{a} = \sqrt{\frac{200}{772}}$ $\times \log.$

$$\times \log. \frac{200 + \sqrt{200^2 - 2^2}}{2} = \sqrt{\frac{100}{386}} \times \mathrm{l.}\ \frac{100 + \sqrt{9999}}{1}$$
$= 2{\cdot}69676$ ſeconds, the whole time when the chain juſt quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the laſt prob. are the ſame; the diſtance deſcended 99 being the ſame in both. For, although the weight l moved in this latter caſe, be double of what it was in the former, the moving force x is alſo double, becauſe here the one end of the chain ſhortens as much as the other end lengthens, ſo that the ſpace deſcended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the ſame in both; and of courſe the velocity and time the ſame for the ſame diſtance deſcended.

PROBLEM XIX.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, paſſing freely over a Tack or a Pulley, while the leſs Weight is drawn up to it by the Deſcent of the heavier Weight at the other End.



Suppoſe the motion to commence at equal diſtances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;
$b = 39\frac{1}{8}$ inc. or $3\frac{25}{96}$ feet, the ſecond's pend.
x = Bw = BW, any ſpace paſſed over;
z = the number of vibrations;

Then $\frac{W - w}{W + w} = f = \frac{1}{3}$ is the accelerating force. And hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$. But, by the nature of pendulums, $\sqrt{a \pm x} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a \pm x}}$ the vibrations per ſecond made by either weight, namely, the longer or ſhorter, according as the upper or under ſign is uſed, if the threads were to continue of that length for 1 ſecond. Hence, then, as

$$1'' : \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{ax \pm x^2}},$$

the fluxion of the number of vibrations.

Now when the upper ſign + takes place, the fluent is $z = 2\sqrt{\frac{b}{4gf}} \times \log. \frac{\sqrt{x} + \sqrt{a + x}}{\sqrt{a}} = \sqrt{\frac{b}{4gf}} \times \log. \frac{a + 2x + 2\sqrt{ax + x^2}}{a}$. And when $x = a$, the ſame then becomes $z = \sqrt{\frac{b}{gf}} \times \log.\ 1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times \log.\ 1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times \log.\ 1 + \sqrt{2} = {\cdot}688511$, the whole number of vibrations made by the deſcending weight.

But when the lower ſign, or −, takes place, the fluent is $\sqrt{\frac{b}{4gf}} \times$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p\sqrt{\frac{b}{gf}} = 3{\cdot}1416 \times \sqrt{\frac{3 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3{\cdot}1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}} = 1{\cdot}227091$, the whole number of vibrations made by the leſſer or aſcending weight.

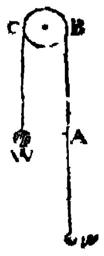
SCHOL. It is evident that the whole number of vibrations, in each caſe, is the ſame, whatever the length of the thread is. And that the greater number is to the leſs, as 1·5708 to hyp. log. $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the ſame time t, by an invariable pendulum, conſtantly of the ſame length a, is $\sqrt{\frac{b}{gf}} = \cdot 781190$. For the time of deſcending the ſpace a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 ſecond; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the conſtant number of vibrations.

So that the three numbers of vibrations, namely, of the aſcending, conſtant, and deſcending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137; whatever be the length of the thread.

PROBLEM XX.

To determine the Circumſtances of the Aſcent and Deſcent of two unequal Weights, ſuſpended at the two Ends of a Thread paſſing over a Pulley: the Weight of the Thread and of the Pulley being conſidered in the Solution. Let



l = the whole length of the thread;
a = the weight of the ſame;
b = Aw the dif. of lengths at firſt;
d = W − w the dif. of the two weights;
c = a wt. applied to the circumference, ſuch as to be equal to its whole wt. and friction reduced to the circumference;
s = W + w + a + c the ſum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at firſt. But if x denote any variable ſpace deſcended by W, or aſcended by w, the difference of the lengths of the thread will be altered $2x$; ſo that the difference will then be $b - 2x$, and its weight $\frac{b - 2x}{l}a$; conſequently the motive force there will be $d - \frac{b - 2x}{l}a$ $= \frac{dl - ab + 2ax}{l}$, and theref. $\frac{dl - ab + 2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = 2gf\dot{x} =$ $2g\dot{x} \times \frac{dl - ab + 2ax}{sl}$; the fluents of which give $v^2 = 4gx \times \frac{dl - ab + 2ax}{sl}$, or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{ex + x^2}$

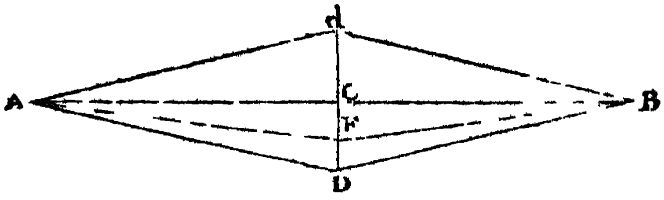
the general expreſſion for the velocity, putting $e = \frac{dl - ab}{a}$. And when $x = b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Alſo, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{s}}$, as it ought.

Again, for the time, $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{ex + x^2}}$; the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x} + \sqrt{e + x}}{\sqrt{e}}$ the general expreſſion for the time of deſcending any ſpace x.

And if the radicals be expanded in a ſeries, and the log. of it be taken, the ſame time will become $t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl - ab}} \times (1 - \frac{x}{6e} + \frac{3x^2}{40e^2}$ &c) Which, therefore, becomes barely $\sqrt{\frac{sx}{dg}}$ when a, the weight of the thread, is nothing; as it ought.

PROBLEM XXI.

To find the Velocity and Time of Vibration of a ſmall Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and ſtretched by a given Tenſion; the Extent of the Vibration being very ſmall.



Let l = AC half the length of the thread;
a = CD the extent of the vibration;
x = CE any variable diſtance from C;
w = wt. of the ſmall body fixed to the middle;
W = a wt. which, hung at each end of the thread, will be equal to the conſtant tenſion at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE :: W the force in direction EA : the force in direction EC. Or, becauſe AC is nearly = AE, the vibration being very ſmall; taking AC inſtead of AE, it is AC : CE :: W : $\frac{Wx}{l}$ the force in EC ariſing from the tenſion in EA. Which will be alſo the ſame for that in EB. Therefore the ſum is $\frac{2Wx}{l}$ = the whole motive force in EC ariſing from the tenſions on both ſides. Conſequently $\frac{2Wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions is $v\dot{v}$ or $2gf\dot{s} = \frac{-4g\text{w}x\dot{x}}{lw}$; and the fluents $v^2 = -\frac{4g\text{w}x^2}{lw}$. But when $x = a$, this is $-\frac{4g\text{w}a^2}{lw}$, and ſhould be $= 0$; therefore the correct fluents are $v^2 = 4g\text{w} \times \frac{a^2 - x^2}{lw}$, and $v = \sqrt{4g\text{w} \times \frac{a^2 - x^2}{lw}}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{g\text{w}}{lw}}$ for the greateſt velocity at the point C.

Now if we ſuppoſe $w = 1$ grain, $\text{w} = 5$ lb troy, or 28800 grains, and $2l = \text{AB} = 3$ feet; the velocity C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{3}a$. So that
if $a = \frac{1}{10}$ inc. the greateſt veloc. is $9\frac{1}{4}$ ft. per ſec.
if $a = 1$ inc. the greateſt veloc. is $92\frac{37}{60}$ ft. per ſec.
if $a = 6$ inc. the greateſt veloc. is $555\frac{7}{10}$ ft per ſec.
To find the time t, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{\text{w}g}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}}$

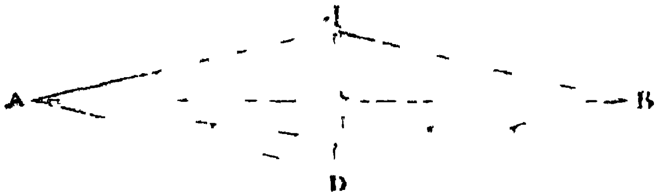
Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{\text{w}g}} \times$ arc to coſine $\frac{x}{a}$ and radius 1, the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration, is $\cdot7854\sqrt{\frac{wl}{\text{w}g}}$; and conſequently the time of a whole vibration through Dd is $1\cdot5708\sqrt{\frac{wl}{\text{w}g}}$.

Uſing the foregoing numbers, namely $w = 1$, $\text{w} = 28800$, and $2l = 3$ feet; this expreſſion for the time is $\frac{1111\frac{2}{3}}{3\cdot1416} = 353\frac{5}{9}$, the number of vibrations per ſecond. But if $w = 2$, there would be 250 vibrations per ſecond; and if $w = 100$, there would be $35\frac{16}{45}$ vibrations per ſecond.

PRO-

PROBLEM XXII.

To determine the ſame as in the laſt Problem, when the Diſtance CD bears ſome ſenſible Proportion to the Length AB; the Tenſion of the Thread however being ſtill ſuppoſed a Conſtant Quantity.



Uſing here the ſame notation as in the laſt problem, and taking the true variable length AE for AC, it is AE or EB : CE :: 2 W : $\frac{2\text{W}x}{\text{AE}} = \frac{2\text{W}x}{\sqrt{l^2 + x^2}}$ the whole motive force from the two equal tenſions W in AE and EB; and therefore $\frac{2\text{W}}{w} \times \frac{x}{\sqrt{l^2 + x^2}} = f$ is the accelerative force at E. Therefore the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4\text{W}g}{w} \times \frac{-x\dot{x}}{\sqrt{l^2 + x^2}}$; and the fluents $v^2 = \frac{8\text{W}g}{w} \times -\sqrt{l^2 + x^2}$. But when $x = a$, theſe are $0 = \frac{8\text{W}g}{w} \times -\sqrt{l^2 + a^2}$; therefore the correct fluents are $v^2 = \frac{8\text{W}g}{w} \times (\sqrt{l + a^2} - \sqrt{l^2 + x^2}) = \frac{8\text{W}g}{w} \times (\text{AD} - \text{AE})$. And hence $v = \sqrt{\frac{8\text{W}g}{w} \times (\text{AD} - \text{AE})}$ the general expreſſion for the velocity at E. And when E arrives at C, it gives the greateſt velocity there $= \sqrt{\frac{8\text{W}g}{w} \times (\text{AD} - \text{AC})}$.

Which, when $W = 28800$, $w = 1$, $2l = 3$ feet, and $CD = 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10} - 3}{2}}$ $= 548\frac{1}{3}$ feet per ſecond. Which came out $555\frac{7}{10}$ in the laſt problem, by uſing always AC for AE in the value of f. But when the extent of the vibrations is very ſmall, as $\frac{1}{10}$ of an inch, as it commonly is, this greateſt velocity here will be $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{1}{43200}} =$ $9\frac{1}{2}$ nearly, which in the laſt problem was $9\frac{1}{4}$.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8Wg}} \times \frac{-\dot{x}}{\sqrt{c - \sqrt{l^2 + x^2}}}$, making $c = AD = \sqrt{l^2 + a^2}$.—To find the fluent the eaſier, multiply the numerator and denominator both by $\sqrt{c + \sqrt{l^2 + x^2}}$, ſo ſhall $\dot{t} = \sqrt{\frac{w}{8Wg}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}} \times \sqrt{c + \sqrt{l^2 + x^2}}$. Expand now the quantity $\sqrt{c + \sqrt{l^2 + x^2}}$ in a ſeries, and put $d = c + l$, ſo ſhall $\dot{t} = \sqrt{\frac{wd}{8Wg}} \times (1 + \frac{x^2}{4dl} - \frac{2d + l}{32d^2l^3}x^4 +$ $\frac{4d^2 + 2dl + l^2}{128d^3l^5}x^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^4l^7}x^8$ &c.)

Now the fluent of the firſt term $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ is $=$ the arc to ſine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q be the fluents of any other two ſucceſſive terms, without the coefficients, the diſtance of Q from the firſt term $\dot{A}$ being n; then it is evident that $\dot{Q} = x^2\dot{P} = x^{2n}\dot{A}$, and $\dot{P} = x^{2n-2}\dot{A}$. Aſſume therefore $Q = bP - ex^{2n-1}$ $\sqrt{a^2 - x^2}$; then is $\dot{Q}$ or $x^2\dot{P} = b\dot{P} - \overline{2n - 1} \cdot ex^{2n-2}\dot{x}$

$$\sqrt{a^2 - x^2} + \frac{ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} = b\dot{P} - \frac{(2n-1)\,ea^2x^{2n-2}\dot{x}}{\sqrt{a^2 - x^2}}$$

$$+ \frac{(2n-1)\,ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} + \frac{ex^{2n}\dot{x}}{\sqrt{a^2 - x^2}} = b\dot{P} - (2n-1)\,ea^2\dot{P}$$

$$+ (2n-1)\,ex^2\dot{P} + ex^2\dot{P} = b\dot{P} - (2n-1)\,ea^2\dot{P} +$$

$2nex^2\dot{P}$. Then, comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n-1)\,ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$. Conſequently $Q = \frac{(2n-1)\,a^2P - x^{2n-1}\sqrt{a^2 - x^2}}{2n}$, the general equation between any two ſucceſſive terms, and by means of which the ſeries may be continued as far as we pleaſe. And hence, neglecting the coefficients, putting A = the firſt term, namely the arc whoſe ſine is $\frac{x}{a}$, and B, C, D, &c, the following terms, the ſeries is as follows,

$$A + \frac{a^2A - \surd x\overline{a^2 - x^2}}{2} + \frac{3a^2B - x^3\sqrt{a^2 - x^2}}{4} +$$

$\frac{5a^2C - x^5\sqrt{a^2 - x^2}}{6}$ &c. Now when $x = 0$, this ſeries $= 0$; and when $x = a$, the ſeries becomes $\frac{1}{2}p + \frac{a^2A}{2} + \frac{3a^2B}{4} + \frac{5a^2C}{6}$ &c, where $p = 3{\cdot}1416$, or the ſeries is $\frac{1}{2}p\,(1 + \frac{1}{2}a^2 + \frac{1\cdot3}{2\cdot4}a^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}a^6$ &c.)

So that, by taking in the coefficients, the general time of paſſing over any diſtance DE will be $\sqrt{\frac{w\,(c+l)}{8wg}} \times$

$$\tfrac{1}{2}p \times (1 + \frac{1}{4dl}\cdot\tfrac{1}{2}a^2 \qquad - \frac{2d+l}{32d^2l^3}\cdot\frac{1\cdot3}{2\cdot4}a^4 \text{ \&c,}$$

$$-\text{arc sin}\,\frac{x}{a} - \frac{1}{4dl}\cdot\frac{a^2A - x\sqrt{a^2 - x^2}}{2} + \frac{2d+l}{32d^2l^3}\cdot\frac{3a^2B - x^3\sqrt{a^2 - x^2}}{4}$$

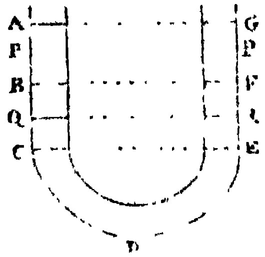
&c.

And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of paſſing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w\,(c+l)}{2\,\mathrm{w}g}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d+l}{32d^2l^3} \cdot \frac{1\cdot 3}{2\cdot 4}a^4 + \frac{4d^2+2dl+l^2}{128d^3l^5} \cdot \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}a^6 - \frac{40d^3+8d^2l+12dl^2+5l^3}{2048d^4l^7} \cdot \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}a^8\,\&c.)$ Which, when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{\mathrm{w}g}}$, the ſame as in the laſt problem, as it ought.

Taking here the ſame numbers as in the laſt problem, viz. $l = \frac{3}{2}$, $a = \frac{1}{2}$, $w = 2$, $\mathrm{w} = 28800$, $g = 16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w\,(c+l)}{2\,\mathrm{w}g}} = \cdot 0040514$, and the ſeries is $1 + \cdot 006762 - \cdot 000175 + \cdot 000003\ \&c = 1\cdot 006590$; therefore $\cdot 0040514 \times 1\cdot 006590 = \cdot 0040965 = \frac{1}{245\frac{1}{5}}$ is the time of one whole vibration, and conſequently $245\frac{1}{5}$ vibrations are performed in a ſecond: which were 250 in the laſt problem.

PROBLEM XXIII.

It is propoſed to determine the Velocity, and the Time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any poſition whatever, the whole being of a uniform diameter or width throughout. Let water, or quickſilver, or any other fluid, be poured in, till it ſtand in equilibrio, at any horizontal line BF. Then let one ſurface be preſſed or puſhed down by ſhaking, from B to C, and the other will aſcend through the equal ſpace FG; after which let them be permitted freely to return. The ſurfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oſcillations are therefore required.

When the ſurfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each ſide, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt. of 2PF}}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and a = FG or BC; then is $\frac{a}{l}$ the accelerative force. Putting therefore x = GP any variable diſtance, v the velocity, and t the time; then PF $= a - x$, and $\frac{2a - 2x}{l} = f$ the accelerative force; hence $v\dot{v}$ or $2gf\dot{s}$ $= \frac{4g}{l}(a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{4g}{l}$ $(2ax - x^2)$, and $v = \sqrt{4g \times \frac{2ax - x^2}{l}}$ is the general expreſſion for the velocity at any term. And when $x = a$, it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greateſt velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{2ax - x^2}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$ arc to verſed ſine $\frac{x}{a}$ and radius 1, the general expreſſion for the time. And when $x = a$, it becomes $t = \frac{1}{4}p\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being $= 3\cdot1416$; and conſequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the ſame, whatever AB is, the whole length l remaining the ſame.

And the time of vibration is alſo equal to the time of the vibration of a pendulum whoſe length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that if the length l be $78\frac{1}{4}$ inches, it will oſcillate in 1 ſecond.

SCHOL. This reciprocation of the water in the canal, is nearly ſimilar to the motion of the waves of the ſea. For the time of vibration is the ſame, however ſhort the branches are, provided the whole length be the ſame. So that when the height is ſmall, in proportion to the length of the canal, the motion is ſimilar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a ſpace nearly equal to its breadth, in the time of two vibrations of a pendulum whoſe length is ($\frac{1}{2}l$) half the length of the canal, or one fourth of the breadth of a wave, or in the time of one vibration of a pendulum whoſe length is the whole breadth of the wave, ſince the times of vibration are as the ſquare roots of their lengths. Conſequently, waves whoſe breadth is equal to 39⅛ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a ſecond, or 195⅝ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or leſs waves will be increaſed or diminiſhed in the ſubduplicate ratio of their breadths.

PROBLEM XXIV.

To determine the Time of filling the Ditches of a Work with Water at the Top, by a Sluice of 2 Feet ſquare; the Head of Water above the Sluice being 10 Feet, and the Dimenſions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the ſluice, the area of which is 4 ſquare feet; therefore $8\sqrt{10g}$ is the quantity per ſecond running through it; and conſequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31'\ 3''$ nearly is the time filling the ditch.

PROBLEM XXV.

To determine the Time of emptying a Veſſel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom, the Height of the Aperture being very ſmall in reſpect of the Altitude of the Fluid.

Put a = the area of the aperture or ſluice;
$2g = 32\frac{1}{6}$ feet, the force of gravity;
d = the whole depth of water;
x = the variable alt. of the ſurface above the aperture;
A = the area of the ſurface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will iſſue at the ſluice; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$ the velocity with which the ſurface of the water will deſcend at the altitude x, or the ſpace it would deſcend in 1 ſecond with the velocity there. Now in deſcending the ſpace $\dot{x}$, the velocity may be conſidered as uniform; and uniform deſcents are as their times; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of deſcending $\dot{x}$ ſpace, or the fluxion of the time of exhauſting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}$.

Then when the nature or figure of the veſſel is given, there will be given A in terms of x; which value of A being ſubſtituted into this fluxion of the time, the fluent of the reſult will be the time of exhauſting ſought.

So if, for example, the veſſel be any priſm, or every where of the ſame breadth; then A is a conſtant quantity,

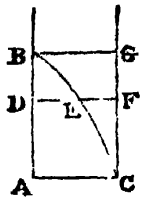
tity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and ſhould be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the ſurface deſcending till the depth of the water be x. And when $x = 0$, the whole time of exhauſting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

And hence if A be 10000 ſquare feet, $a = 1$ ſquare foot, and $d = 10$ feet; the time is $7885\frac{1}{3}$ ſeconds, or $2^h\ 11'\ 25''\frac{1}{3}$.

Again, if the veſſel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $99 : 90 + x :: 22 : \frac{90 + x}{99} \times 22$ the breadth of the ſurface of the water when its depth in the canal is x; and conſequently $A = \frac{90 + x}{99} \times 22000$ is the ſurface at that time. Conſequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{gx}} = 11000 \times \frac{90 + x}{99} \times \frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $11000 \times \frac{180 + \frac{2}{3}d}{99a} \times \sqrt{\frac{d}{g}} = \frac{11000 \times 186 \times 3}{99 \times 4\frac{1}{95}} = 15459''\frac{2}{3}$ nearly, or $4^h\ 17'\ 39''\frac{2}{3}$, the whole time of exhauſting by a ſluice of 1 foot ſquare.

PROBLEM XXVI.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.



Let AB $= x$ the variable height of water at any time;
AC $= b$ the breadth of the cut;
$d =$ the whole or firſt depth of water;
A $=$ the area of the ſurface of the water in the ditch;
$g = 16\frac{1}{12}$ feet.

The velocity at any point D, is as $\sqrt{\text{BD}}$, that is, as the ordinate DE of a parabola BEC, whoſe baſe is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Conſequently the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom in the ſame time, as the area of the parabola ABC, to the area of the paralellogram ABGC, that is as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity at AC; therefore $\frac{2}{3} \times 2\sqrt{gx} \times bx = \frac{4}{3} bx\sqrt{gx}$ is the quantity diſcharged per ſecond through ABGC; and conſequently $\frac{4bx\sqrt{gx}}{3\text{A}}$ is the velocity per ſecond of the deſcending ſurface. Hence then $\frac{4bx\sqrt{gx}}{3\text{A}} : -\dot{x} :: 1'' : \frac{-3\text{A}\dot{x}}{4bx\sqrt{gx}} = \dot{t}$ the fluxion of the time of deſcending.

Now when A the ſurface of the water is conſtant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{ax}}$ for the general time of ſinking the ſurface to any depth x. And when $x = 0$, this expreſſion is infinite; which ſhews that the time of a complete exhauſtion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $\text{A} = 21 \times 1000 = 21000$, and it be required to exhauſt the water down to $\frac{1}{16}$ of a foot deep; then $x = \frac{1}{16}$, and the above expreſſion becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{96}} \times \frac{3 - \frac{1}{4}}{\frac{3}{4}} = 14400''$, or juſt 4 hours for that time. And if it be required to depreſs it 8 feet, or till 1 foot depth of water remain in the ditch, the time of ſinking the water to that point will be $43'\ 38''$.

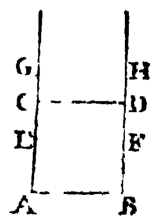
Again, if the ditch be the ſame depth and length as before, but 20 feet broad at bottom, and 22 at top; then the deſcending ſurface will be a variable quantity, and, by prob. 25, it will be $= \frac{90 + x}{99} \times 22000$; hence in this caſe the fluxion of the time, or $\frac{-3\text{A}\dot{x}}{4bx\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90 + x}{x\sqrt{x}}\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times \left(\frac{90 - x}{\sqrt{x}} - \frac{90 - d}{\sqrt{d}}\right)$ for the time of ſinking the water to any depth x.

Now when $x = 0$, this expreſſion for the complete exhauſtion becomes infinite.

But if $x = 1$ foot, the time t is $42'\ 56''\frac{1}{2}$.
And when $x = \frac{1}{16}$ foot, the time is $3^h\ 50'\ 28''\frac{1}{2}$.

PROBLEM XXVII.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet ſquare, the Bottom of which is level with the Bottom of the Ditch, and the Height of the ſupplying Water is 9 Feet above the Bottom of the Ditch.



Let ACDB repreſent the area of the vertical ſluice, being a ſquare of 9 ſquare feet, and AB level with the bottom of the ditch. And ſuppoſe the ditch filled to any height AF, the ſurface being then at FE.

Put $a = 9$ the height of the head or ſupply,
$b = 3 = \text{AB} = \text{AC}$,
$g = 16\frac{1}{12}$,
$\text{A} =$ the area of a horizontal ſection of the ditches.
$x = a - \text{AF}$, the height of the head above EF.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water preſſes through the part AEFB; and therefore $2\sqrt{gx} \times \text{AEFB} = 2b\sqrt{gx}\,(a - x)$ is the quantity per ſecond running through AEFB. Alſo, the quantity running per ſecond through ECDF is $2\sqrt{gx} \times \frac{11}{12}\,\text{ECDF} = \frac{11}{6}b\sqrt{gx}\,(b - a + x)$ nearly. For the real quantity is, by proceeding as in the laſt prob. the dif. between two parab. ſegs. the alt. of the one being x, its baſe b, and the alt. of the other $a - b$; and the medium of that dif. between its greateſt ſtate at AB, where it is $\frac{9}{10}$ AD, and its leaſt ſtate at CD, where it is 0, is nearly $\frac{11}{12}$ ED. Conſequently the ſum of the two, or $\frac{1}{6}b\sqrt{gx}\,(a + 11b - x)$ is the quantity per ſecond running in by the whole ſluice ACDB.

Hence

Hence then $\frac{1}{6}b\sqrt{gx} \times \frac{a + 11b - x}{\text{A}} = v$ is the rate for velocity per ſecond with which the water riſes in the ditches; and ſo $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6\text{A}}{b\sqrt{g}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c - x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a conſtant quantity, and in that caſe the correct fluent of this fluxion is $t = \frac{6\text{A}}{b\sqrt{gc}} \times \log.\left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \times \frac{\sqrt{c} - \sqrt{x}}{\sqrt{c} + \sqrt{x}}\right)$ the general expreſſion for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the ſluice. And when $x = \text{AC} = a - b = d$ ſuppoſe, then $t = \frac{6\text{A}}{b\sqrt{gc}} \times \log.\left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \cdot \frac{\sqrt{c} - \sqrt{d}}{\sqrt{c} + \sqrt{d}}\right)$ is the time of filling to CD the top of the ſluice.

Again, for filling to any height GH above the ſluice, x denoting as before $a - \text{AG}$ the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole ſluice AD: and therefore $2b^2\sqrt{gx}$ the quantity per ſecond, and $\frac{2b^2\sqrt{gx}}{\text{A}} = v$ the riſe per ſecond of the water in the ditches; conſequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-\text{A}}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being 0 when $x = a - b = d$, is $t = \frac{\text{A}}{b^2\sqrt{g}}(\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the ſum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{\text{A}}{b\sqrt{g}}\left[\frac{\sqrt{d} - \sqrt{x}}{b} + \frac{6}{\sqrt{c}} \log.\left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \cdot \frac{\sqrt{c} - \sqrt{d}}{\sqrt{c} + \sqrt{d}}\right)\right]$

for the whole time required. And, uſing the numbers in the problem, this becomes $\frac{A}{3\sqrt{g}} [\frac{\sqrt{6} - \sqrt{3}}{3} + \frac{6}{\sqrt{42}} \times \log. (\frac{\sqrt{42} + \sqrt{9}}{\sqrt{42} - \sqrt{9}} . \frac{\sqrt{42} - \sqrt{6}}{\sqrt{42} + \sqrt{6}})] = 0{\cdot}03577277 A$, the time in terms of A the area of the length and breadth, or horizontal ſection of the ditches. And if we ſuppoſe that area to be 200000 ſquare feet, the time required will be 7154″, or $1^h\ 54'\ 14''$.

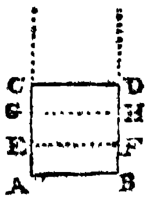
And if the ſides of the ditch ſlope a little, ſo as to be a little narrower at the bottom than at top, the proceſs will be nearly the ſame, ſubſtituting for A its variable value, as in prob. 25 and 26. And the time of filling will be very nearly the ſame as that above determined.

PROBLEM XXVIII.

But if the Water, from which the Ditches are to be filled, be the Tide, which at Low Water is below the Bottom of the Trunk, and riſes to 9 Feet above the Bottom of it by a regular Riſe of One Foot in Half an Hour; it is required to aſcertain the Time of Filling it to 6 Feet high, as before in the laſt Problem.

Let ACDB repreſent the ſluice; and when the tide has riſen to any height GH, below CD the top of the ſluice, without the ditches, let EF be the mean height of the water within. And put

$b = 3 = AB = AC$;
$g = 16\frac{1}{12}$;
A = horizontal ſection of the ditches;
$x = AG$;
$z = AE$.



Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x - z)}$ the veloc. of the water through AEFB; and

$\sqrt{g} : \sqrt{\text{EG}} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x - z)}$ the mean veloc. thro' FGHF; theref. $2bz\sqrt{g(x - z)}$ is the quant. per ſec. thro' AEFB; and $\frac{4}{3}b(x - z)\sqrt{g(x - z)}$ is the ſame through EGHF; conſeq. $\frac{2}{3}b\sqrt{g} \times (2x + z)\sqrt{x - z}$ is the whole through AGHB per ſec. This quantity divided by the ſurface A, gives $\frac{2b\sqrt{g}}{3\text{A}} \times (2x + z)\sqrt{x - z} = v$ the velocity per ſecond with which EF, or the ſurface of the water in the ditches, riſes. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x + z)\sqrt{x - z}}.$$

But, as GH riſes uniformly 1 foot in 30′ or 1800″, theref. $1 : \text{AG} :: 1800'' : 1800x = t$ the time of the tide riſing thro' AG; conſeq. $\dot{t} = 1800\dot{x} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x + z)\sqrt{x - z}}$, or $m\dot{z} = (2x + z)\sqrt{x - z} \cdot \dot{x}$ is the fluxional equation expreſſing the relation between x and z; when $m = \frac{\text{A}}{1200b\sqrt{g}} = \frac{3200}{231}$ or $13\frac{197}{231}$ when $\text{A} = 200000$ ſquare feet.

Now to find the fluent of this equation, aſſume $z = \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{8}{2}} + \text{C}x^{\frac{11}{2}} + \text{D}x^{\frac{14}{2}}$ &c. So ſhall

$$\sqrt{x - z} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{4}{2}} - \frac{\text{A}^2 + 4\text{B}}{8}x^{\frac{7}{2}} - \frac{\text{A}^3 + 4\text{AB} + 8\text{C}}{16}x^{\frac{10}{2}} \text{ \&c,}$$

$$2x + z = 2x + \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{8}{2}} + \text{C}x^{\frac{11}{2}} \text{ \&c,}$$

$$(2x + z)\sqrt{x - z}\,\dot{x} = 2x^{\frac{3}{2}}\dot{x} * - \frac{3\text{A}^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{\text{A}^3 + 6\text{AB}}{4}x^{\frac{12}{2}}\dot{x} \text{ \&c,}$$

and $m\dot{z} = \frac{5}{2}m\text{A}x^{\frac{3}{2}}\dot{x} + \frac{8}{2}m\text{B}x^{\frac{6}{2}}\dot{x} + \frac{11}{2}m\text{C}x^{\frac{9}{2}}\dot{x} + \frac{14}{2}m\text{D}x^{\frac{12}{2}}\dot{x}$ &c.

Then equate the coefficients of the like terms,

ſo ſhall and conſequently

$$\tfrac{5}{2}m\text{A} = 2, \qquad \text{A} = \frac{4}{5m},$$

$$\tfrac{3}{2}m\text{B} = 0, \qquad \text{B} = 0,$$

$$\tfrac{11}{2}m\text{C} = -\tfrac{3}{4}\text{A}^2, \qquad \text{C} = -\frac{24}{275m^3},$$

$$\tfrac{14}{2}m\text{D} = -\tfrac{1}{4}\text{A}^3 - \tfrac{3}{2}\text{AB}, \qquad \text{D} = -\frac{16}{875m^4},$$

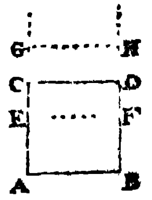
&c ; &c.

Then theſe values of A, B, C, &c, ſubſtituted in the aſſumed value of z, give

$$z = \frac{4}{5m}x^{\frac{5}{2}} * - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \text{ \&c ;}$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 = \text{AC}$, than $z = \cdot886$ of a foot, or $10\frac{2}{3}$ inches, = AE, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the firſt hour and half of time.



Again, to find the time, after the above, when FF arrives at CD, or when the water in the ditches arrives as high as the top of the ſluice.

The notation remaining as before,

then $2bz\sqrt{g\,(x-z)}$ per ſec. runs thro' AF,

and $\frac{8}{5}b\,(3 - z)\sqrt{g\,(x-z)}$ per ſec. thro' ED nearly;

theref. $\frac{2}{5}b\sqrt{g} \times (12 + z)\sqrt{x-z}$ is the whole per ſecond through AD nearly.

confeq. $\dfrac{2b\sqrt{g}}{5\text{A}} \times (12+z)\sqrt{x-z} = v$ is the velocity per ſecond of the point E; and therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{5\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(12+z)\sqrt{x-z}}$$

$= 1800\dot{x}$, or $m\dot{z} = (12+z)\sqrt{x-z}\cdot\dot{x}$, where $m = \dfrac{\text{A}}{720b\sqrt{g}} = 23\frac{2}{13}$ nearly.

Aſſume $z = Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}} + Dx^{\frac{6}{2}}$ &c. So ſhall $\sqrt{x - z} = x^{\frac{1}{2}} - \frac{A}{2}x^{\frac{2}{2}} - \frac{A^2 + 4B}{8}x^{\frac{3}{2}} - \frac{A^3 + 4AB + 8C}{16}x$ &c; $12 + z = 12 + Ax^{\frac{3}{2}} + Bx^{\frac{4}{2}} + Cx^{\frac{5}{2}}$ &c; $(12 + z) \cdot \sqrt{x - z} \cdot \dot{x} = 12x^{\frac{1}{2}}\dot{x} - 6Ax^{\frac{2}{2}}\dot{x} - (\frac{1}{2}A^2 + 6B)x^{\frac{3}{2}}\dot{x}$ &c; $m\dot{z} = \frac{3}{2}mAx^{\frac{1}{2}}\dot{x} + \frac{4}{2}mBx^{\frac{2}{2}}\dot{x} + \frac{5}{2}mCx^{\frac{3}{2}}\dot{x}$ &c.

Then, equating the like terms, &c, we have

$$A = \frac{\;}{m},\ B = -\frac{24}{m^2},\ C = \frac{96}{5m^3},\ D = \frac{64}{3m^2} \text{ nearly, \&c.}$$

$$\text{Hence } z = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 + \frac{96}{5m^3}x^{\frac{5}{2}} + \frac{64}{3m^2}x^3 \text{ \&c.}$$

$$\text{Or } z = \frac{8}{m}x^{\frac{3}{2}} \text{ nearly.}$$

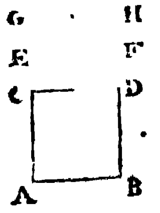
But, by the firſt proceſs, when $x = 3$, $z = \cdot 886$; which ſubſtituted for them, we have $z = \cdot 886$, and the ſeries $= 1 \cdot 63$; therefore the correct fluents are

$$z - \cdot 886 = -1 \cdot 63 + \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c,}$$

$$\text{or } z + \cdot 744 = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c.}$$

And when $z = 3 =$ AC, it gives $x = 6.369$ for the height of the tide without, when the ditches are filled to the top of the ſluice, or 3 feet high; which anſwers to $3^{h}\ 11'\ 4''$.

Laſtly, to find the time of riſing the remaining 3 feet above the top of the ſluice; let



$x =$ CG the height of the tide above CD,
$z =$ CE ditto in the ditches above CD;
and the other dimenſions as before.

Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)}$ = the velocity with which the water runs through the whole ſluice AD; conſequently AD $\times\ 2\sqrt{g\ (x-z)} = 18\sqrt{g\ (x-z)}$ is the quantity per ſecond running through the ſluice, and $\frac{18\sqrt{g}}{\text{A}}\sqrt{x-z} = v$ the velocity of z, or the riſe of the water in the ditches, per ſecond; hence $v : \dot{z} :: 1'' : \dot{t} =$

$$\frac{\dot{z}}{v} = \frac{\text{A}}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{x-z}} = 1800\dot{x}, \text{ and } n\dot{z} = \dot{x}\sqrt{x-z}$$

is the fluxional equation; where $n = \frac{\text{A}}{180^2\sqrt{g}} = \frac{3200}{2079}$.

To find the fluent,

Aſſume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c.

Then $x - z = x - \text{A}x^{\frac{3}{2}} - \text{B}x^{\frac{4}{2}} - \text{C}x^{\frac{5}{2}}$ &c,

$$\dot{x}\sqrt{x-z} = x^{\frac{1}{2}}\dot{x} - \frac{\text{A}}{2}x^{\frac{2}{2}}\dot{x} - \frac{\text{A}^2 + 4\text{B}}{8}x^{\frac{3}{2}}\dot{x} \text{ \&c,}$$

$$n\dot{z} = \tfrac{3}{2}n\text{A}x^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}n\text{B}x^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}n\text{C}x^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

Then equating the like terms gives

$$\text{A} = \frac{2}{3n},\ \text{B} = \frac{-1}{6n^2},\ \text{C} = \frac{1}{90n^3},\ \text{D} = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3$ &c.

But, by the ſecond caſe, when $z = 0$, $x = 3{\cdot}369$, which being uſed in the ſeries, it is $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2$ &c.

And when $z = 3$, $x = 7$; the heights above the top of the ſluice; anſwering to 6 and 10 feet above the bottom of the ditches. That is, for the water to riſe to the height of 6 feet within the ditches, it is neceſſary for the tide to riſe to 10 feet without, which juſt anſwers to 5 hours ; and ſo long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 ſquare feet.

Moreover, when $x = 6$, then $z = 2{\cdot}117$ the height above the top of the ſluice ; to which add 3, the height of the ſluice, and the ſum 5·117, is the depth of water in the ditches in 4 hours and a half, or when the tide has riſen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of water, it is taken for granted that the velocity is the ſame as that which is due to the whole height of the ſurface of the ſupplying water. A ſuppoſition which agrees with the principles of the greater number of authors: though ſome make the velocity to be that which is due to the half height only: and others make it ſtill leſs.

Alſo in ſome places, where the difference between two parabolic ſegments was to be taken, in eſtimating the mean velocity of the water through a variable orifice, I have uſed a near mean value of the expreſſion ; which makes the operation of finding the fluents much more eaſy, and is at the ſame time ſufficiently exact for the purpoſe in hand.

OF THE

MOTION OF BODIES IN FLUIDS.

PROBLEM XXIX.

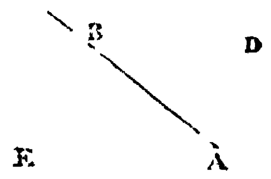
To determine the Force of Fluids in Motion. And the Circumſtances attending Bodies moving in Fluids.

1. It is evident that the reſiſtance to a plane, moving perpendicularly through an infinite fluid, at reſt, is equal to the preſſure or force of the fluid upon the plane at reſt, and the fluid moving with the ſame velocity, and in the contrary direction, to that of the plane in the former caſe. But the force of the fluid in motion, muſt be equal to the weight or preſſure which generates that motion; and which, it is known, is equal to the weight or preſſure of a column of the fluid, whoſe baſe is equal to the plane, and its altitude equal to the height through which a body muſt fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the ſake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the ſpecific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole reſiſtance, or motive force m, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{12}$ feet. And hence, *cæteris paribus*, the reſiſtance is as the ſquare of the velocity.

2. This ratio, of the ſquare of the velocity, may be otherwiſe derived thus. The force of the fluid in motion, muſt be as the force of one particle multiplied by

the number of them: but the force of a particle is as its velocity; and the number that ſtrikes the plane in a given time, is alſo as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the ſquare of the velocity.

3. If the direction of motion, inſtead of being perpendicular to the plane, as above ſuppoſed, were inclined to it in any angle, the ſine of that angle being s, to the radius 1; then the reſiſtance to the plane, or the force of the fluid againſt the plane, in the direction of the motion, as aſſigned above, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination, or in the ratio of 1 to s^3. For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whoſe ſine is s; the number of particles, or quantity of the fluid, ſtriking the plane, will be diminiſhed in the ratio of 1 to s, or of radius to the ſine of the angle B of inclination; and the force of each particle will alſo be diminiſhed in the ſame ratio of 1 to s; ſo that, on both theſe accounts, the whole reſiſtance will be diminiſhed in the ratio of 1 to s^2, or in the duplicate ratio of radius to the ſine of the ſaid angle. But again, it is to be conſidered that this whole reſiſtance is excited in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all theſe accounts, the reſiſtance in the direction of motion, is diminiſhed in the ratio of 1 to s^3, or in the triplicate ratio of radius to the ſine of inclination. Hence, comparing this with article 1, the whole reſiſtance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{4g}$.



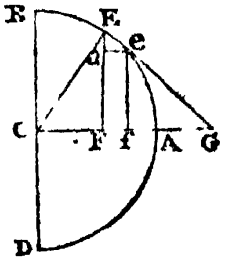
4. Alſo if w denote the weight of the body, whoſe plane face a is reſiſted by the abſolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

5. And if the body be a cylinder, whoſe face or end is a, and diameter d, or radius r, moving in the direction of its axis; becauſe then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the reſiſting force m will be $\frac{npd^2v^2}{16g} = \frac{npr^2v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw}$ $= \frac{npr^2v^2}{4gw}$.

6. This is the value of the reſiſtance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical ſurface, or an elliptic ſection, or any other figure every where equally inclined to the axis, the ſine of inclination being s: then, the number of particles of the fluid ſtriking the face being ſtill the ſame, but the force of each, oppoſed to the direction of motion, diminiſhed in the duplicate ratio of radius to the ſine of inclination, the reſiſting force m will be $\frac{npd^2v^2s^2}{16g} = \frac{npr^2v^2s^2}{4g}$. But if the body were terminated by an end or face of any other form, as a ſpherical one, or ſuch like, where every part of it has a different inclination to the axis, then a farther inveſtigation becomes neceſſary, ſuch as in the following propoſition.

PROBLEM XXX.

To determine the Reſiſtance of a Fluid to any Body, moving in it, of a Curved End, as a Sphere, or a Cylinder with a Hemiſpherical End, &c.



1. Let BEAD be a ſection through the axis CA of the ſolid, moving in the direction of that axis. To any point of the curve draw the tangent FG, meeting the axis produced in G: alſo draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG. Putting CF $= x$, EF $= y$, BE $= z$, $s =$ ſine ∠ G to radius 1, and $p = 3 \cdot 1416$; then $2py$ is the circumference whoſe radius is CF, or the circumference deſcribed by the point E, in revolving about the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the ſurface, or the ſurface deſcribed by Ee, in the ſaid revolution about CA, and which is the quantity repreſented by a in art. 3 of the laſt problem: hence $\frac{nv^2s^3}{4g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{2g} \times y\dot{z}$ is the reſiſtance on that ring, or the fluxion of the reſiſtance to the body, whatever the figure of it may be. And the fluent of which will be the reſiſtance required.

2. In the caſe of a ſpherical form, putting the radius CA or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$ or EF × Ee = CE × ae $= r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3y\dot{z}$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{2gr^2} \times x^3\dot{x}$; the fluent of which, or $\frac{pnv^2}{8gr^2}x^4$, is the re-

reſiſtance to the ſpherical ſurface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the reſiſtance on the whole hemiſphere; which is alſo equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular reſiſtance to the circle of the ſame diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, ſhews that the reſiſtance to the ſphere, is juſt equal to half the direct reſiſtance to a great circle of it, or to a cylinder of the ſame diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe, if N denote its denſity or ſpecific gravity, its weight w will be $= \frac{1}{6}pd^3\text{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{p\text{N}d^3} = \frac{3nv^2}{16g\text{N}d}$ by art. 8 of the general theorems in page 169; hence then $\frac{3n}{4\text{N}d} = \frac{1}{s}$, and $s = \frac{\text{N}}{n} \times \frac{4}{3}d$, which is the ſpace that would be deſcribed by the globe, while its whole motion is generated or deſtroyed by a conſtant force which is equal to the force of reſiſtance, if no other force acted on the globe to continue its motion. And if the denſity of the fluid were equal to that of the globe, the reſiſting force is ſuch, as, acting conſtantly on the globe without any other force, would generate or deſtroy its motion in deſcribing the ſpace $\frac{4}{3}d$, by that accelerating or retarding force.

5. Hence the greateſt velocity that a globe will acquire by deſcending in a fluid, by means of its relative weight in the fluid, will be found by making the reſiſt-

ing force equal to that weight. For after the velocity is arrived at ſuch a degree, that the reſiſting force is equal to the weight that urges it, it will increaſe no longer, and the globe will afterwards continue to deſcend with that velocity uniformly. Now, N and n being the ſeparate ſpecific gravities of the globe and fluid, $N - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3(N - n)$ the weight by which it is urged; alſo $m = \frac{pnv^2d^2}{32g}$ is the reſiſtance; conſequently $\frac{pnv^2d^2}{32g} = \frac{1}{6}pd^3(N - n)$ when the velocity becomes uniform; from which is found $v = \sqrt{4g \cdot \frac{4}{3}d \cdot \frac{N - n}{n}}$ for the ſaid uniform or greateſt velocity.

And by comparing this form with that in art. 6 of the general theorems in page 169, it will appear that its greateſt velocity is equal to the velocity generated by the accelerating force $\frac{N - n}{n}$ in deſcribing the ſpace $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely deſcribing the ſpace $\frac{N - n}{n} \times \frac{4}{3}d$.——If $N = 2n$, or the ſpecific gravity of the globe be double that of the fluid, then $\frac{N - n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greateſt velocity in deſcribing $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter.——It is farther evident that if the body be very ſmall, it will very ſoon acquire its greateſt velocity, whatever its denſity may be.

Ex. If a leaden ball, of 1 inch diameter, deſcend in water, and in air of the ſame denſity as at the earth's ſurface, the three ſpecific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193}$ $= 8{\cdot}5944$ feet, is the greateſt velocity per ſecond the ball can acquire by deſcending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$ is the greateſt velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greateſt velocities it could acquire, would be only $\frac{1}{10}$ of theſe, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were ſtill farther diminiſhed, the greateſt velocity would alſo be diminiſhed, and that in the ſubduplicate ratio of the diameter of the ball.

PROBLEM XXXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid in which it is projected with a given Velocity.

1. Let $a =$ the firſt velocity of projection, x the ſpace deſcribed in any time t, and v the velocity then. Now, by art. 4 of the laſt problem, the accelerative force $f = \frac{3nv^2}{16g\text{N}d}$, where N is the denſity of the ball, n that of the fluid, and d the diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8\text{N}d}\dot{x}$, and hence $\frac{\dot{v}}{v} = \frac{-3n}{8\text{N}d}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8\text{N}d}$. And the correct fluent of this is log. a — log. v or log. $\frac{a}{v} = bx$. Or, put $c = 2{\cdot}718281828$, the number

whoſe hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the firſt velocity, therefore the velocity loſt in any time will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the firſt velocity.

Ex. 1. If a globe be projected, with any velocity, in a medium of the ſame denſity with itſelf, and it deſcribe a ſpace equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and the velocity loſt is $\frac{c^{bx} - 1}{c^{bx}} = \frac{2\cdot08}{3\cdot08}$, or nearly $\frac{2}{3}$ of the projectile velocity.

Ex. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per ſecond, to find the velocity loſt after moving through 500 feet of air; we ſhould have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $N = 7\frac{1}{2}$, $n = \cdot0012$; and therefore $bx = \frac{3nx}{8Nd} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 993$ feet per ſecond: having loſt 207 feet, or nearly $\frac{1}{6}$ of its firſt velocity.

Ex. 3. If the earth revolved about the ſun in a medium as denſe as the atmoſphere near the earth's ſurface, and it were required to find the quantity of motion loſt in a year. Then, ſince the earth's mean denſity is about $4\frac{1}{2}$, and its diſtance from the ſun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx = \frac{3 \cdot 75398 \cdot 12 \cdot 2}{8 \cdot 10000 \cdot 9} = 7{\cdot}5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{25575}{25576}$ parts are loſt of the firſt motion in the ſpace of a year, and only the $\frac{1}{25576}$ part remains.

Ex. 4. If it be required to determine the diſtance moved, x, when the globe has loſt any part of its motion, as ſuppoſe $\frac{1}{2}$, and the denſity of the globe and fluid equal: The general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times$ log. of $2 = 1{\cdot}8483925d$. So that the globe loſes half its motion before it has deſcribed twice its diameter.

3. To find the time t, we have $\dot{t} = \frac{s}{v} = \frac{x}{v} = \frac{c^{bx}x}{a}$. Now to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conſequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$, and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vaniſh together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab} = \frac{1}{ab}$, therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time ſought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Ex. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per ſecond; and it were required to determine in what time it would paſs over 500 yards or 1500 feet, and what would be its velocity at the end of that time: We ſhould have, as in ex. 2 above, $b = \frac{3 \cdot 12 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1\cdot7372}{1200} = \frac{1}{690}$ nearly. Conſequently $v = 690$ is the velocity, and $t = \frac{1}{b}\left(\frac{1}{v} - \frac{1}{a}\right) = 2716 \times \left(\frac{1}{690} - \frac{1}{1200}\right) = 1\frac{31}{46}$ ſeconds is the time required, or 1″ and $\frac{2}{3}$ nearly.

PROBLEM XXXII.

To determine the Relations of Space, Time, and Velocity, when a Globe deſcends, by its own Weight, in an infinite Fluid.

The foregoing notation remaining, viz. d = diameter, N and n the denſity of the ball and fluid, and v, s, t, the velocity, ſpace, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3(\text{N} - n)$ = its weight in the fluid, alſo $m = \frac{pnd^2v^2}{32g}$ = its reſiſtance from the fluid; conſequently $\frac{1}{6}pd^3(\text{N} - n) - \frac{pnd^2v^2}{32g}$ is the force by which the ball is urged; which being divided by $\frac{1}{6}p\text{N}d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{\text{N}} - \frac{3nv^2}{16g\text{N}d}$ for the accelerative force.

2. Hence $v\dot{v} = 2gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{\text{N}vv}{2g(\text{N} - n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8\text{N}d}$, and $\frac{1}{a} = \frac{3n}{2g \cdot 8d(\text{N} - n)}$, or $ab = 2g$; the fluent of which is $s = \frac{1}{2b} \times \log. \frac{a}{a - v^2}$, an expreſſion for the ſpace s in terms of the velocity v.

3. But now to determine v in terms of s, put $c = 2{\cdot}718281828$; then ſince $\log. \frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; and hence $v = \sqrt{a - ac^{-2bs}}$ the velocity ſought.

4. The greateſt velocity is to be found, as in art. 5 of prob. 30, by making f or $1 - \frac{n}{\text{N}} - \frac{3nv^2}{16g\text{N}d} = 0$, which gives $v = \sqrt{2g \cdot 8d \cdot \frac{\text{N} - n}{3n}} = \sqrt{a}$. The ſame value is alſo obtained by making the fluxion of v^2 or $a - ac^{-2bs} = 0$. And the ſame value of v is obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident, however, that it will approximate towards it the faſter, the greater b is, or the leſs d is; and that, the diameters being very ſmall, the bodies deſcend by nearly uniform velocities, which are directly in the ſubduplicate ratio of the diameters. See alſo art. 5 prob. 30 for other obſervations on this head.

5. Since c^{-2bs} is the number whoſe log is $-2bs$, it will be $c^{-2bs} = c^{-e} = 1 - e + \frac{1}{2}e^2 - \frac{1}{2 \cdot 3}e^3 + \frac{1}{2 \cdot 3 \cdot 4}e^4$ &c, putting $e = 2bs = \frac{3ns}{4Nd}$; hence $1 - c^{-e} = e - \frac{1}{2}e^2 + \frac{1}{6}e^3 -$ &c, and $v = \sqrt{a} \times \sqrt{1 - c^{-e}} = \sqrt{ae} \times (1 - \frac{1}{4}e + \frac{5}{96}e^2 - \frac{1}{128}e^3$ &c). And when N is very great in reſpect of n, then, all the terms after the firſt being very ſmall, v will be nearly $= \sqrt{ae} = \sqrt{4gs \cdot \frac{N-n}{n}} = \sqrt{4gs}$ nearly, that is the velocity freely generated by gravity, as it ought.

6. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$; conſequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + z}{1 - z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expreſſion for the time.

7. When N is very great in reſpect of n; then, as in art. 5, $v = \sqrt{2abs}$, and $\log. \frac{\sqrt{a} + v}{\sqrt{a} - v} = \log. \frac{\sqrt{a} + \sqrt{2abs}}{\sqrt{a} - \sqrt{2abs}} = \log. \frac{1 + \sqrt{2bs}}{1 - \sqrt{2bs}} = 2\sqrt{2bs}$; and therefore $t = \frac{2\sqrt{2bs}}{2b\sqrt{a}} = \sqrt{\frac{2s}{ab}} = \sqrt{\frac{s}{g}}$, the ſame as the time of deſcending freely by gravity, as it ought.

Ex. If it were required to determine the time and velocity, by deſcending in air 1000 feet; the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{12} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} = \frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2}$;

conſeq. $v = \sqrt{a} \times \sqrt{1 - c^{-2bs}} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times \sqrt{1 - c^{-\frac{81}{85}}} = 203\frac{2}{3}$ the velocity. And $t = \frac{}{2b\sqrt{a}}$ $\log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \sqrt{\frac{34 \cdot 2500}{27 \cdot 193}} \times \log. \frac{1 \cdot 78383}{0 \cdot 21617} =$ $8 \cdot 5236''$, the time.

Note. If the globe be ſo light as to aſcend in the fluid; it is only neceſſary to change the ſigns of the firſt two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all reſpects as before.

SCHOLIUM.

To compare this theory, contained in the laſt four problems, with experiment, I ſhall here extract the few following numbers from extenſive tables of velocities and reſiſtances, reſulting from a courſe of many hundred very accurate experiments, made by me in the courſe of the year 1786; of which a particular account will be given elſewhere.

In the firſt column are contained the mean uniform or greateſt velocities acquired in air, by globes, hemiſpheres, cylinders, and cones, all of the ſame diameter, and the altitude of the cone nearly equal to the diameter alſo, when urged by the ſeveral weights, expreſſed in avordupois ounces, and ſtanding on the ſame line with the velocities, each in their proper columns. So, in the firſt line, the numbers ſhew that, when the greateſt or uniform velocity was accurately 3 feet per ſecond, the bodies were urged by theſe weights, according as their different ends went foremoſt; namely, by ·028oz when the vertex of the cone went foremoſt; by ·064oz when the baſe of the cone went foremoſt; by ·027oz for a whole ſphere; by ·045oz for a cylinder; by ·051oz for the flat ſide of the hemiſphere; and by ·020oz for the round or convex ſide of the hemiſphere. Alſo, at the bottom of all, are placed the mean proportions of the reſiſtances of theſe figures, in the neareſt whole numbers. Note, the common diameter of all the figures, was 6·375 or $6\frac{3}{8}$ inches; ſo that the area of the circle of that diameter is juſt 32 ſquare inches, or $\frac{2}{9}$ of a ſquare foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Alſo the diameter of the ſmall hemiſphere was $4\frac{3}{4}$ inches, and conſequently the area of its baſe is $17\frac{3}{4}$ ſquare inches, or $\frac{1}{8}$ of a ſquare foot nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; and conſequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz nearly in thoſe circumſtances.

Velocity per ſec.	Cone		Whole globe	Cylinder	Hemiſphere		Small Hemiſ.
	vertex	baſe			flat	round	flat
feet	oz	oz	oz	oz	oz	oz	oz
3	·028	·064	·027	·045	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·292	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of reſiſtances, ſeveral practical inferences may be drawn. As,

1. That the reſiſtance is nearly as the ſurface; the reſiſtance increaſing but a very little above that proportion in the greater ſurfaces. Thus, by comparing together the numbers in the 6th and laſt columns, for the baſes of the two hemiſpheres, the areas of which are in the proportion of 17¼ to 32, or as 5 to 9 very nearly; but the numbers in thoſe two columns, expreſſing the reſiſtances are nearly as 1 to 2, or as 5 to 10, as far as to the velocity

of 12 feet; after which the reſiſtances on the greater ſurface increaſe gradually more and more above that proportion. And the mean reſiſtances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumſtance therefore agrees nearly with the theory.

2. The reſiſtance to the ſame ſurface, is nearly as the ſquare of the velocity; but gradually increaſes more and more above that proportion as the velocity increaſes. This is manifeſt from all the columns. And therefore this circumſtance alſo nearly agrees with the theory, in ſmall velocities.

3. When the hinder parts of bodies are of different forms, the reſiſtances are different, though the fore parts be alike; owing probably to the different preſſures of the air on the hinder parts. Thus, the reſiſtance to the fore part of the cylinder is leſs than that on the flat baſe of the hemiſphere, or of the cone; becauſe the hinder part of the cylinder is more preſſed or puſhed, by the following air, than thoſe of the other two figures.

4. The reſiſtance on the baſe of the hemiſphere, is to that on the convex ſide, nearly as $2\frac{2}{5}$ to 1, inſtead of 2 to 1, as the theory aſſigns the proportion. And therefore in this particular, the theory is attended with a conſiderable error.

5. The reſiſtance on the baſe of the cone, is to that on the vertex, nearly as $2\frac{3}{10}$ to 1, inſtead of $5\frac{1}{4}$ to 1, as the theory in pa. 223 art. 6 requires it to be. So that the theory in this inſtance gives leſs than half the true experimented reſiſtance.

6. Hence we can find the altitude of a column of air, whoſe preſſure ſhall be equal to the reſiſtance of a body, moving through it with any velocity.

Let a = the area of the ſection of the body, ſimilar to any of thoſe in the table, perpendicular to the direction of motion;
r = the reſiſtance to the velocity in the table; and
x = the altitude ſought, of a column of air, whoſe baſe is a, and its preſſure r.

Then ax = the content of the column in feet,
and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces;
therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude ſought in feet, namely $\frac{5}{6}$ of the quotient of the reſiſtance of any body divided by its tranſverſe ſection; which is a conſtant quantity for all ſimilar bodies, however different in magnitude, ſince the reſiſtance r is as the ſection a, as we have found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in our table, except the ſmall hemiſphere; then $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the reſiſtance in our table to the ſimilar body. If, for example, we take the convex ſide of the large hemiſphere, whoſe reſiſtance is ·634 oz to a velocity of 16 feet per ſecond; then r = ·634, and $x = \frac{15}{4}r = 2{\cdot}3775$ feet, is the altitude of the column of air whoſe preſſure is equal to the reſiſtance on a ſpherical ſurface with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$ the altitude due to the velocity 16; which is near double the altitude that is equal to the preſſure. And as the altitude is proportional to the ſquare of the velocity, therefore, in ſmall velocities, the reſiſtance to any ſpherical ſurface, is equal to the preſſure of a column of air on its great circle, whoſe altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

7. Hence we may infer the great reſiſtance ſuffered by military projectiles. For we find in the table that a

globe of $6\frac{3}{8}$ inches diameter, which is equal to the ſize of an iron ball weighing 36 lb, moving with a velocity of only 16 feet per ſecond, meets with a reſiſtance equal to the preſſure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the ſquare of the velocity, the leaſt reſiſtance that ſuch a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the preſſure of 417 lb, and that independent of the preſſure of the atmoſphere itſelf on the fore part of the ball, which would be 480 lb more, as there would be no preſſure from the atmoſphere on the hinder part, in the caſe of ſo great a velocity as 1600 feet per ſecond. So that the whole reſiſtance would not be leſs than about 900 lb to ſuch a velocity.

8. Having ſaid in the laſt article that the preſſure of the atmoſphere is taken intirely off the hinder part of the ball, moving with a velocity of 1600 feet per ſecond; which muſt happen when the ball moves faſter than the particles of air can follow by ruſhing into the place quitted and left void by the ball, or when the ball moves faſter than the air ruſhes into a vacuum from the preſſure of the incumbent air. Let us therefore inquire what this velocity is. Now the velocity with which any fluid iſſues, depends upon its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, ſuppoſing the height of the barometer to be 30 inches or $2\frac{1}{2}$ feet, the height of an uniform atmoſphere, all of the ſame denſity as at the earth's ſurface, would be $2\frac{1}{2} \times 14 \times 833\frac{1}{3}$ or 29166 feet; therefore $\sqrt{16} : \sqrt{29166} :: 32 : 8\sqrt{29166} = 1366$ feet, which is the velocity ſought. And therefore with a velocity of 1600 feet per ſecond, or any velocity above 1366 feet, the ball muſt continually leave a vacuum behind it, and ſo muſt ſuſtain the whole preſſure of the

moſphere on its fore part, as well as the reſiſtance iſing from the vis inertia of the particles of air ſtruck the ball.

9. Upon the whole we find th[illegible]he reſiſtance of the air, determined by our experiments, differs very widely, th in reſpect to the quantity of it on all figures, and in ſpect to the proportions of it on oblique ſurfaces, from e ſame as determined by the preceding theory, which is e ſame as that of Sir Iſaac Newton, and moſt modern ilofophers. Neither ſhould we ſucceed better if we ve recourſe to the theory given by profeſſor Graveſande, others, as ſimilar differences and inconſiſtencies ſtill cur.

We conclude therefore that all the theories of the eſiſtance of the air hitherto given, are very erroneous. And I have only laid down the preceding one, till further xperiments on this important ſubject ſhall enable us to educe from them, another, that ſhall be more conſonant the true phenomena of nature.

F I N I S.

ERRATA.

Note *b*, at the line, denotes counted from the bottom.

Pa.	Line.	Correction.
2	8	*for* baſe *r.* ſide
3, 4		In the fig. of C ſet A for B at the axis
6	2*b*	2CD . CF
7	13	*for* BG *read* FG
8	7*b*	LIK
9	8*b*	*for* one *read* c
13	13	ſtraight
[illegible]	4	ſemi-axes
	7	or DG . 2IK :: AD + AD . AG : ID²
22	7	proportional
25		the fig. is turned
56	12	draw the line AG in the fig.
62		ſet [illegible] in the fig.
68	4	Hyperbola
69	7*b*	inſcribed between the hyperbolas
[illegible]	9*b*	$CG^2 -$ C[illegible]
[illegible]	[illegible]	aſymptote
[illegible]	5	*for* E *read* F
[illegible]		In the fig. H and L to change places
[illegible]		Draw the line AIE in the fig.
122		ſet M and K in the fig.
123	4	EK . KL
134	2*b*	firſt col. *for* 12 *read* 10
140	10	*for* 8lb *read* 6lb.

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